Workshop on Computability Theory 2016
Ghent, Belgium — July 4 & 5, 2016

Schedule

The workshop is held in the *Faculteit Letteren en Wijsbegeerte* building at Blandijnberg 2. The talks are in Auditorium A, and the coffee breaks are in room 100.017.

Monday, July 4

Before 9:50am:	Coffee available! (in room 100.017)
9:50am – 10:00am:	Welcome to Ghent!
10:00am – 10:30am:	Laurent Bienvenu *Diagonally non-computable functions and fireworks*
10:40am – 11:10am:	Alexandra Soskova *Structural properties of spectra and co-spectra*
11:20am – 11:50am:	Lars Kristiansen *On subrecursive representability of irrational numbers*
12:00pm – 2:00pm:	Lunch!
2:00pm – 2:30pm:	Liesbeth De Mol *"Beyond" Turing computability: a historical perspective*
2:40pm – 3:10pm:	Sam Sanders *The unreasonable effectiveness of nonstandard analysis*
3:20pm – 3:50pm:	Coffee break! (in room 100.017)
3:50pm – 4:20pm:	Ludovic Patey *How randomly rainbows appear!*
4:30pm – 5:00pm:	Andrea Sorbi *Computably enumerable equivalence relations under computable reducibility*

Tuesday, July 5

Before 10:00am:	Coffee available! (in room 100.017)
10:00am – 10:30am:	Uri Andrews *Spectra of recursive models of trichotomous strongly minimal theories*
10:40am – 11:10am:	Matthew Harrison-Trainor *Computable structures of high Scott rank*
11:20am – 11:50am:	Valentina Harizanov *Limitwise monotonic functions and equivalence structures*
12:00pm – 2:00pm:	Lunch!
2:00pm – 2:30pm:	Takayuki Kihara *The structure of natural many-one degrees*
2:40pm – 3:10pm:	Noah Schweber *Computability-theoretic aspects of ordinals*
3:20pm – 3:50pm:	Coffee break! (in room 100.017)
3:50pm – 4:20pm:	Jeffry Hirst *Counting uses of Ramsey's theorem*

Abstracts (alphabetical by author)

Uri Andrews (University of Wisconsin–Madison)

Spectra of recursive models of trichotomous strongly minimal theories

In recursive model theory, it is natural to explore the situation where some models of a theory are recursive while others are not. We focus on strongly minimal theories which form model-theoretic building blocks for more complex theories, and offer a nice dimension-theory for structures. The spectrum of a theory is the set of dimensions of computable models of that theory. I will share what is known about which sets are spectra of recursive models for structures which are algebraic in nature, formalized by the Zilber trichotomy.

Laurent Bienvenu (CNRS, l'Université Montpellier)

Diagonally non-computable functions and fireworks

Fireworks arguments are a particular type of probabilistic algorithm in computability theory, first developed by Kautz to show that almost every real (in the sense of Lebesgue measure) computes a 1-generic. We show how to combine such arguments with bushy tree forcing to prove a number of results about diagonally non-computable degrees. (This is joint work with Ludovic Patey.)

Liesbeth De Mol (CNRS, Université de Lille 3)

"Beyond" Turing computability: a historical perspective

In this talk I will review the positions of Church and Post on computability, contrast them with Turing's and explain why their models have had an important impact on early programming practices (again, in contrast to Turing's).

Valentina Harizanov (George Washington University)

Limitwise monotonic functions and equivalence structures

In recent years, limitwise monotonic functions and sets and their generalizations have been playing an increasingly important role in computable algebra and computable model theory. The notion of a limitwise monotonic function was introduced by N. Khisamiev in 1981 in his study of computable properties of abelian p-groups. This notion captures the dynamic enumeration of the elements in the sets being formed. We will present results involving applications of limitwise monotonic functions to investigate computability-theoretic properties of countable equivalence structures. Most recent work is joint with E. Fokina and D. Turetsky.

Matthew Harrison-Trainor (University of California, Berkeley)

Computable structures of high Scott rank

The Scott rank of a countable structure measures the complexity of characterizing that among all countable structures. A computable structure has Scott rank at most $\omega_1^{CK}+1$; if it has Scott rank ω_1^{CK} or $\omega_1^{CK}+1$, we say that it has high Scott rank. There are a small number of known examples of models of high Scott rank. The Harrison linear order is one example, and Knight and Millar re-worked a construction of Makkai to build a computable structure of Scott rank ω_1^{CK}. Calvert, Knight, and Millar later gave another example of Scott rank ω_1^{CK}.

I will talk about two new examples of structures of high Scott rank, with properties different from the known examples. A structure $\mathcal{A}$ of high Scott rank is said to be computably approximable if every computable formula φ true in $\mathcal{A}$ is true in a model of low Scott rank. We give the first known examples of models of high Scott rank which are not computably approximable.

The two previously known examples of computable models of Scott rank ω_1^{CK} had the property that their computable infinitary theory is $\aleph_0$-categorical. Millar and Sacks asked whether this is always the case. They were able to construct a structure $\mathcal{A}$ whose computable infinitary theory was not $\aleph_0$-categorical, but $\mathcal{A}$ was not computable: it satisfied $\omega_1^{\mathcal{A}}=\omega_1^{CK}$. In joint work with Igusa and Knight, we produce a computable such structure.

Jeffry Hirst (Appalachian State University)

Counting uses of Ramsey's theorem

Suppose we write $\mathsf{RT}(2,n)$ for Ramsey's theorem for pairs and n colors. In this talk, we address the question: Can $\mathsf{RT}(2,4)$ be proved with one use of $\mathsf{RT}(2,2)$? Of course, the answer depends on the base system chosen and the formalization of what constitutes a use. In some settings, a formalization of Weihrauch reductions can lend insight, and the law of the excluded middle plays a surprising role.

Takayuki Kihara (University of California, Berkeley)

The structure of natural many-one degrees

Under $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}$, we show that, in a certain sense, the structure of "natural" many-one degrees is isomorphic to the Wadge degrees. Indeed, if $\mathcal{Q}$ is a very strong better-quasi-order, the same holds true for $\mathcal{Q}$-valued many-one/Wadge degrees (e.g., many-one/Wadge degrees of k-partitions, k-coverings, ordinal-valued maps, etc.)

Formally, in this talk, a "natural" problem A is supposed to be *relativizable* and *uniformly degree invariant*, that is, if given two oracles X and Y are Turing equivalent, then the relativized problems A^X and A^Y are many-one equivalent, and moreover, one can effectively obtain a witness of $A^X \equiv_m A^Y$ from a witness of $X \equiv_T Y$. For instance, the halting problem (the Turing jump operation) and its (transfinite) iterations, the hyperjump operation, the sharp operation are all natural. Given uniformly degree invariant relativizable problems A, B, we say that *A is many-one-on-a-cone reducible to B* if there is an oracle C such that for any $X \geq_T C$, A^X is many-one reducible to B^X relative to C. We show that there is a natural one-to-one correspondence between the many-one-on-a-cone degrees of uniformly degree invariant relativizable decision problems ($\mathcal{Q}$-valued problems, resp.) and the Wadge degrees of subsets of Baire space ($\mathcal{Q}$-valued functions on Baire space, resp.)

This is joint work with Antonio Montalbán.

Lars Kristiansen (Universitetet i Oslo)

On subrecursive representability of irrational numbers

We consider various ways to represent irrational numbers by subrecursive functions: via Cauchy sequences, Dedekind cuts, trace functions, several variants of sum approximations and continued fractions. Let $\mathcal{S}$ be a class of subrecursive functions. The set of irrational numbers that can be obtained with functions from $\mathcal{S}$ depends on the representation. We compare the sets obtained by the different representations.

A function $C : \mathbb{N} \to \mathbb{Q}$ is a *Cauchy sequence* for the real number α when $|\alpha - C(n)| < 1/2^n$. A function $D : \mathbb{Q} \to \{0, 1\}$ is a *Dedekind cut* of the real number α when $D(q) = 0$ iff $q < \alpha$. A function $T : \mathbb{Q} \to \mathbb{Q}$ is a *trace function* for the irrational number α when $|\alpha - q| > |\alpha - T(q)|$.

Any irrational number α can be written of the form $\alpha = a + \frac{1}{2^{k_0}} + \frac{1}{2^{k_1}} + \frac{1}{2^{k_2}} + \dots$ where $k_0, k_1, k_2, \dots$ is a strictly monotone increasing sequence of natural numbers and a is an integer. Let $A : \mathbb{N} \to \mathbb{N}$ be a strictly monotone function. We will say that A is a *sum approximation from below* of the real number α if there exists $a \in \mathbb{Z}$ such that $\alpha = a + \sum_{i=0}^{\infty} 1/2^{A(i)+1}$. Any real number can also be written as a difference between an integer and an infinite sum, and we will say that A is a *sum approximation from above* of the real number α if there exists $a \in \mathbb{Z}$ such that $\alpha = a - \sum_{i=0}^{\infty} 1/2^{A(i)+1}$.

The sum approximations defined above are sum approximations in base 2. We will also consider *general sum approximations* (from above and below). A *general sum approximation* of α is a function that yields the sum approximation of α in any base.

An irrational number α can also be represented by a function $f : \mathbb{N} \to \mathbb{Z}$ where $f(n)$ yields the n^{th} element of the continued fraction $[a_0; a_1, a_2 \dots]$ of α.

Let $\mathcal{P}_C$, $\mathcal{P}_D$ and $\mathcal{P}_{[]}$ denote the sets of irrationals that are representable, respectively, by primitive recursive Cauchy sequences, primitive recursive Dedekind cuts and primitive recursive continued fractions. Specker [3] proved $\mathcal{P}_D \subset \mathcal{P}_C$, and Lehman [2] proved $\mathcal{P}_{[]} \subset \mathcal{P}_D$ (strict inclusions). We will discuss a number of theorems on how trace functions and (general) sum approximation (from above and below) relate to Cauchy sequences, Dedekind cuts and continued fractions. Most of these theorems can be found in Kristiansen [1].

[1] L. Kristiansen, *On subrecursive representability of irrational numbers.* Accepted for publication in Computability (the journal of CiE).

[2] R. S. Lehman, On Primitive Recursive Real Numbers, *Fundamenta Mathematica* **49**(2) (1961), 105–118.

[3] E. Specker, Nicht Konstruktiv Beweisbare Satze Der Analysis, *The Journal of Symbolic Logic* **14**(3) (1949), 145–158.

Ludovic Patey (Université Paris Diderot–Paris 7)

How randomly rainbows appear!

The rainbow Ramsey theorem for pairs asserts that every k-coloring of $[\mathbb{N}]^2$ has an infinite rainbow, that is, an infinite set H such that $[H]^2$ uses each color at most once. Csima and Mileti first gave a probabilistic algorithm to construct a rainbow given a computable coloring. Mileti later gave a surprising characterization of the rainbow Ramsey theorem for pairs in terms of algorithmic randomness. In this talk, we present the computable analysis and the reverse mathematics of the rainbow Ramsey theorem and investigate weakenings of the rainbow Ramsey theorem, stressing the amount of randomness needed to build rainbows.

Sam Sanders (Ludwig-Maximilians-Universität München, Universiteit Gent)

The unreasonable effectiveness of nonstandard analysis

As suggested by the title, the aim of this paper is to uncover the vast computational content of classical Nonstandard Analysis. To this end, we formulate a template which converts a theorem of 'pure' Nonstandard Analysis, i.e. formulated solely with the nonstandard definitions (of continuity, integration, differentiability, convergence, compactness, et cetera), into the associated effective theorem. The latter constitutes a theorem of computable mathematics no longer involving Nonstandard Analysis. We discuss applications of this template in Reverse Mathematics and computability theory. Of particular interest is the 'special fan functional', a new variation of Tait's fan functional with very strange computability properties. This is joint work with Dag Normann.

Noah Schweber (University of California, Berkeley)

Computability-theoretic aspects of ordinals

The computable structure theory of ordinals from the perspective of Muchnik reducibility is completely understood: $\alpha \leq_{\mathrm{w}} \beta$ if and only if β is less than the first admissible above α. However, beyond Muchnik reducibility the picture is much more complicated. In this talk, I will discuss two directions: the problem of effectively (relative to an ordinal) listing countable sets of ordinals, and the structure of ordinals under stronger-than-Muchnik reducibilities following work by, and answering a question of, Hamkins and Li.

Andrea Sorbi (Università di Siena)

Computably enumerable equivalence relations under computable reducibility

We review some recent work done in collaboration with Uri Andrews. If R, S are equivalence relations on the set of natural numbers, then R is *computably reducible to* S ($R \leq S$) if there exists a computable function f such that for all x, y, $x \; R \; y$ if and only $f(x) \; S \; f(y)$. We study the class of all computably enumerable equivalence relations (ceers) under $\leq$, and the corresponding degree structure **Ceers**, which is known to be a bounded poset. The review consists of three sections.

(Universal ceers.) A known class of universal ceers (i.e. ceers lying in the greatest element of **Ceers**) is given by the *uniformly effectively inseparable* ceers (i.e. nontrivial ceers partitioning ω into equivalence classes such that any distinct pair of them is effectively inseparable in a uniform way). The previously known largest class of uniformly effectively inseparable ceers was the class of the *uniformly finitely precomplete* ceers, which can be characterized as those ceers which are computably isomorphic to nontrivial computably enumerable extensions of the relation $\sim_{PA}$ of provable equivalence in Peano Arithmetic. Answering a question in [1], we show that there exist uniformly effectively inseparable ceers that are not uniformly finitely precomplete.

(Definability and automorphisms.) We propose a partition of **Ceers** into three classes: **Fin** (the degrees of finite ceers), **Light** (i.e. the degrees of ceers R such that $\mathrm{Id} \leq R$, where Id is the identity ceer), and **Dark** (the remaining ones). We show that **Fin**, **Light**, **Dark** are first order definable in **Ceers**, in the language of partial orders. Moreover, there are continuum many order automorphisms of **Ceers** fixing **Light** (respectively, **Dark**).

(Jump iterations.) Finally, we review some results on transfinite iterations of the halting jump operation on **Ceers**.

[1] U. Andrews, S. Lempp, J. S. Miller, K. M. Ng, L. San Mauro, and A. Sorbi, *Universal computably enumerable equivalence relations*, J. Symbolic Logic **79** (2014), no. 1, 60–88.

Alexandra Soskova (Sofia University)

Structural properties of spectra and co-spectra

We consider the degree spectrum of a countable structure from the point of view of enumeration reducibility. We will give an overview of several structural properties of degree spectra and co-spectra, such as a minimal pair theorem and the existence of quasi-minimal degrees and receive as a corollary some fundamental theorems in enumeration degrees. We will show that every countable ideal of enumeration degrees is a co-spectrum of a structure and if a degree spectrum has a countable base then it has a least enumeration degree. Next we investigate the omega-enumeration co-spectra and show that not every countable ideal of omega-enumeration degrees is an omega-co-spectrum of a structure.

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