# Workshop on Computability Theory

Prague, July 3–4, 2014

## Information

Welcome to Prague!

### Location

All talks will take place in room S4 of the School of Computer Science, Faculty of Mathematics and Physics, Charles University, located at Malostranská 25, Prague 1.

The building should be open on both mornings, but visitors can also ask to be let in at the gatehouse.

### Directions

The meeting venue is on Malostranské náměstí (Little Quarter Square), below the Prague Castle, and just steps from the Charles Bridge, which connects across the river to various tourist destinations like Old Town Square, Josefov, and Wenceslaus Square. All of these can be reached by foot or by metro or tram.

Trams 12, 20, 22, and 57 (night) stop directly across the square from the building, and connect to the Malostranská metro station. (The station can also be reached by foot in about ten minutes, via Tomášská and then Valdštejnská streets.) Tram 22 is the easiest way to get to the Castle by transit, though the more picturesque way is to walk up the hill.

### Refreshments

A restaurant is located on the first basement level of the workshop building. (Please note that the restaurant will only be available until 13:00 on Thursday.) Numerous restaurants, cafés, and pubs are located on and around Malostranské náměstí.

### WiFi

Secure WiFi access is available via `eduroam`, or by accessing the network `MS-KONFERENCE` with password `Akce-2013`.

# Schedule

## Thursday

| | |
|---|---|
| 9:30–10:30 | Greenberg |
| 10:30–10:45 | Break |
| 10:45–11:45 | Csima |
| 11:45–13:45 | Lunch |
| 13:45–14:45 | Vatev |
| 14:45–15:00 | Break |
| 15:00–16:00 | Soskova |

## Friday

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|---|---|
| 9:00–10:00 | Patey |
| 10:00–10:15 | Break |
| 10:15–11:15 | Kreuzer |
| 11:15–11:30 | Break |
| 11:30–12:30 | Brattka |
| 12:30–14:30 | Lunch |
| 14:30–15:30 | Herbert |
| 15:30–15:45 | Break |
| 15:45–16:45 | Westrick |
| 16:45–17:45 | Khan |

*[*] Fokina's talk, originally scheduled for Thursday, cancelled.*

# Abstracts

## Probabilistic Computability and Choice

Vasco Brattka
*Universität der Bundeswehr München*

We study the computational power of randomized computations on infinite objects, such as real numbers. In particular, we introduce the concept of a Las Vegas computable multi-valued function, which is a function that can be computed on a probabilistic Turing machine that receives a random binary sequence as auxiliary input. The machine can take advantage of this random sequence, but it always has to produce a correct result or to stop the computation after finite time if the random advice is not successful. With positive probability the random advice has to be successful. We characterize the class of Las Vegas computable functions in the Weihrauch lattice with the help of probabilistic choice principles and Weak Weak Kőnig's Lemma. Among other things we prove an Independent Choice Theorem that implies that Las Vegas computable functions are closed under composition. In a case study we show that Nash equilibria are Las Vegas computable, while zeros of continuous functions with sign changes cannot be computed on Las Vegas machines. However, we show that the latter problem admits randomized algorithms with weaker failure recognition mechanisms. The last mentioned results can be interpreted such that the Intermediated Value Theorem is reducible to the jump of Weak Weak Kőnig's Lemma, but not to Weak Weak Kőnig's Lemma itself. These examples also demonstrate that Las Vegas computable functions form a proper superclass of the class of computable functions and a proper subclass of the class of non-deterministically computable functions. We also study the impact of specific lower bounds on the success probabilities, which leads to a strict hierarchy of classes. In particular, the classical technique of probability amplification fails for computations on infinite objects. We also investigate the dependency on the underlying probability space. Joint work with Rupert Hölzl and Guido Gherardi.

## Measuring complexities of classes of structures.

Barbara F. Csima
*University of Waterlow*

How do we compare the complexities of various classes of structures? The Turing ordinal of a class of structures, introduced by Jockusch and Soare, is defined in terms of the number of jumps required for coding to be possible. The back-and-forth ordinal, introduced by Montalbán, is defined in terms of $\Sigma_\alpha$-types. The back-and-forth ordinal is (roughly) bounded by the Turing ordinal. We show that, if we do not restrict the allowable classes, the reverse inequality need not hold. Joint work with Carrie Knoll.

## Degree spectra of structures under equivalence relations*

Ekaterina Fokina
*Kurt Gödel Research Center for Mathematical Logic*

Equivalence relations reflect the idea of similarity between mathematical objects. A large body of work in computable model theory is devoted to the study of complexity of isomorphic copies of a given structure. Some work has been done also for structures equimorphic to a given one.

For a countable structure $\mathscr{A}$ its degree spectrum $DgSp(\mathscr{A})$ was defined in [2] and consists of the Turing degrees of all isomorphic copies of $\mathscr{A}$. Degree spectra of structures have been actively studied since then. More recently, the authors of [1] defined the degree spectrum of a theory $T$ to consist of all degrees of countable models of $T$. We suggest to consider the following generalisation of these notions.

**Definition.** The degree spectrum of a countable structure $\mathscr{A}$ with the universe $\omega$ under the equivalence relation $E$ is

$$DgSp(\mathscr{A},E) = \{deg(\mathscr{B}) \mid \mathscr{B} \text{ is } E\text{-equivalent to } \mathscr{A}\}.$$

Then the classical degree spectrum of $\mathscr{A}$ is $DgSp(\mathscr{A},\cong)$, the degree spectrum of $\mathscr{A}$ under isomorphism, while the degree spectra of the theory of $\mathscr{A}$ is $DgSp(\mathscr{A},\equiv)$, the degree spectrum of $\mathscr{A}$ under elementary equivalence.

We consider degree spectra of structures under other equivalence relations, in particular, $\Sigma_n$-equivalence (the $\Sigma_n$-theories of structures coincide). We call $DgSp(\mathscr{A},\equiv_{\Sigma_n})$ the $\Sigma_n$-spectrum of $\mathscr{A}$. We study what collections of degrees are realisable as $\Sigma_n$-spectra, for various $n$. We give several positive and negative examples for various $n \in \omega$. In particular, we show that the union of two cones is never a $\Sigma_1$-spectrum, but is a $\Sigma_n$-spectrum for $n \geq 2$. The same is true for all non-computable degrees.

This is a joint work with Pavel Semukhin and Dan Turetsky.

**References.**

[1] U. Andrews and J. Miller. Spectra of theories and structures. *Proc. Amer. Math. Soc.*, to appear.
[2] L. J. Richter. Degrees of structures. *J. Symbolic Logic*, 46, 723–731, 1981.

## $\Pi^1_1$ equivalence relations

Noam Greenberg
*Victoria University*

Fokina, Friedman, Harizanov, Knight, McCoy and Montalbán showed that isomorphism between computable structures is universal among all $\Sigma^1_1$ equivalence relations on natural numbers. We show that hyperarithmetic isomorphism is universal for $\Pi^1_1$ equivalence relations. Joint work with Dan Turetsky.

## Weak Lowness Notions for Kolmogrov Complexity

Ian Herbert
*National University of Singapore*

The (prefix-free) Kolmogorov complexity of a finite binary string is the length of the shortest description of the string given by some universal decoding machine. This gives rise to some 'standard' lowness notions for reals: $A$ is $K$-trivial if its initial segments have the lowest possible complexity and $A$ is low for $K$ if using $A$ as an oracle does not decrease the complexity of strings by more than a constant factor. We discuss various ways of weakening these notions and the relations between these weakenings. Kolmogorov complexity also induces some reducibility notions that are weaker than Turing reducibility, and we discuss the behavior of these new lowness notions with respect to these reducibilities.

## Lebesgue density and $\Pi^0_1$ classes

Mushfeq Khan
*University of Wisconsin*

A positive density point is a real such that if it is contained in an effectively closed (or $\Pi^0_1$) set of reals, then the set has positive Lebesgue density around that real. A density-one point is defined analogously. We investigate how these properties interact with various forms of computability-theoretic strength. It was shown by Bienvenu, Hölzl, Miller, and Nies that if we restrict our attention to the Martin-Löf random reals, then the positive density points are exactly the reals that do not compute the halting problem. Does anything similar hold on the more general class of positive density points?

For some classes of reals, it is easy to see that the members cannot have minimal Turing degree. For example, if X is Martin-Löf random or if it is 1-generic, then the sequences given by the even and odd bits of X are Turing incomparable, and hence properly Turing below X. This is not necessarily true of a positive density point. Can such a point be of minimal degree?

We answer these questions and others, working toward a more complete picture of how the two density notions behave.

## Measure theory and higher order arithmetic

Alexander Kreuzer
*National University of Singapore*

I will talk about how to lift computability results to conservation results using proof-theoretic tools.

Let $\phi(X)$ be an arithmetic formula defining the set $\{X : \phi(X)\}$ of reals. By Sacks (1969) and Tanaka (1968) it is known that the measure of this set is also arithmetic. We will use this to obtain from this that $ACA_0^\omega$, the higher order extension of $ACA_0$, plus the statement "all set have a Lebesgue measure" is $\Pi^1_2$-conservative over $ACA_0$.

We will discuss similar result for ultrafilters, idempotent ultrafilters and iterated Hindmann's theorem.

**References.**

[1] A. P. Kreuzer. Measure theory and higher order arithmetic. arXiv:1312.1531.

## On universal instances of principles in reverse mathematics

Ludovic Patey
*Université Paris Diderot (VII)*

Most statements of reverse mathematics are of the form

$$(\forall X)(\exists Y)\Phi(X,Y)$$

where $\Phi$ is an arithmetical formula. In this case, $X$ is called an instance and any $Y$ such that $\Phi(X,Y)$ holds is called a solution.

A statement admits a universal instance $U$ if for every instance $X$,every solution to $U$ computes a solution to $X$. A few principles are known to admit a universal instance, e.g., König's lemma, the rainbow Ramsey's theorem. But many others do not have one. This is for example the case of the Ramsey's theorem for pairs and the ascending descending sequence principle.

We will present different proof techniques for proving the absence of universal instances for ranges of principles, classifying almost the whole Zoo of Damir Dzhafarov in terms of admitting universal instance or not.

## A parallel between classical computability theory and effective definability in abstract structures

Alexandra Soskova
*Sofia University*

There is a close parallel between classical computability and the effective definability on abstract structures. For example, the $\Sigma^0_{n+1}$ sets correspond to the sets definable by means of computable infinitary $\Sigma_{n+1}$ formulae on a structure $\mathfrak{A}$. We will present some analogues for abstract structures of Ashs reducibilities between sets of natural numbers and sequences of sets of natural numbers, given by I. Soskov in his last paper [3]. He generalizes the method of Markers extensions for a sequence of structures. I. Soskov demonstrates that for any sequence of structures its Markers extension codes the elements of the sequence so that the $n$-th structure of the sequence appears positively at the $n$-th level of the definability hierarchy.

We apply these results and generalize the notion of degree spectrum with respect to an infinite sequence of structures $\mathfrak{A}$ in two ways as Joint spectra of $\mathfrak{A}$ [1] and Relative spectra of $\mathfrak{A}$ [2]. We study the set of all lower bounds of the generalized notions in terms of enumeration and $\omega$-enumeration reducibility. The results of Soskov provide a general method given a sequence of structures to construct a structure with $n$-th jump spectrum contained in the spectrum of the $n$-th member of the sequence.

As an application a structure with spectrum consisting of the Turing degrees which are non-low$_n$ for all $n < \omega$ is obtained. Soskov shows also an embedding of the $\omega$-enumeration degrees into the Muchnik degrees generated by spectra of structures.

**References.**


[1] Soskova, A. A. and Soskov, I. N. Co-spectra of joint spectra of structures. *Ann. Univ. Sofia*, 96, 35–44, 2004.
[2] Soskova, A. A. Relativized degree spectra. *J. Logic and Computation*, 17, 1215–1234, 2007.
[3] Soskov I. N. Effective properties of Markers Extensions. *J. Logic and Computation*, 23, 1335–1367, 2013.


## Coding a set by a sequence of structures


Stefan Vatev
*Sofia University*


The idea of coding a set by a sequence of structures arises naturally in computable structure theory and can be found in a hidden form in many constructions. It was first studied independently by Ash and Knight [1].

Let $\{\mathscr{B}_0, \mathscr{B}_1\}$ be a pair of computable structures in the same relational language. We say that the set $S \subseteq \omega$ is coded by the sequence of structures $\{C_n\}_{n\in\omega}$ if

$$C_n \cong \begin{cases} \mathscr{B}_1 & \text{if } n \in S, \\ \mathscr{B}_0 & \text{if } n \notin S. \end{cases}$$

If the sequence $\{C_n\}$ is *uniformly computable*, we say that $S$ is *strongly coded* by the pair $\{\mathscr{B}_0, \mathscr{B}_1\}$.

**Theorem 1** (Ash-Knight [1]). Fix a computable successor ordinal $\alpha$. Let $\mathscr{B}_0$ and $\mathscr{B}_1$ be computable structures in the same relational language. Moreover, let the pair $\{\mathscr{B}_0, \mathscr{B}_1\}$ be $\alpha$-friendly, and $\mathscr{B}_0$, $\mathscr{B}_1$ satisfy the same infinitary $\Sigma_\beta$ sentences, for all $\beta < \alpha$. Then every $\Delta^0_\alpha$ set $S$ is strongly coded by $\{\mathscr{B}_0, \mathscr{B}_1\}$.

If we have a way to distinguish $\mathscr{B}_0$ from $\mathscr{B}_1$, we will be able to extract the set $S$ from the sequence $\{C_n\}_{n\in\omega}$.

**Corollary 1.** Let $\alpha$ be a computable successor ordinal. Let $\mathscr{B}_0$ and $\mathscr{B}_1$ be as in Theorem 1, but also require that there be a computable infinitary $\Sigma_\alpha$ sentence $\Phi_i$ true in $\mathscr{B}_i$, but not true in $\mathscr{B}_{1i}$, for $i = 0, 1$. Then a set $S$ is $\Delta^0_\alpha$ if and only if $S$ is strongly coded by $\{\mathscr{B}_0, \mathscr{B}_1\}$.

For a structure $A = (\omega;\ R_1, \ldots, R_m)$, we can code every $k_i$-ary relation $R_i$ by a sequence of structures $\{C^i_n\}_{n\in\omega}$. It is also easy to see that we can represent any infinite sequence $\{C_n\}_{n\in\omega}$ as a single structure $\mathscr{N}$. These observations help us obtain a jump inversion theorem for Turing degree spectra of structures.

**Corollary 2.** Let $\alpha$ be a computable successor ordinal and let $\mathscr{A}$ be a structure such that $Spec(\mathscr{A}) \subseteq \{\mathbf{d} \mid \mathbf{0}^{(\alpha)} \leq \mathbf{d}\}$. Then we can build a structure $\mathscr{N}$ such that $Spec(\mathscr{A}) = Spec_\alpha(\mathscr{N})$.

In what follows, we will see how by relaxing or strengthening the requirements for the pair $\mathscr{B}_0$ and $\mathscr{B}_1$ we can get different results. We will spend some time discussing the following theorem.

**Theorem 2** (Vatev [2]). We can also obtain the result in Corollary 2 by omitting the requirement that $\{\mathscr{B}_0, \mathscr{B}_1\}$ is $\alpha$-friendly.

As another application, we will show that this construction can be applied in the study of categoricity spectra of structures. This notion, introduced in [3], is relatively new and not well studied yet.

**Proposition 1.** Let $\alpha$, $\mathscr{B}_0$, and $\mathscr{B}_1$ be as in Corollary 2, but with the additional requirement that $\mathscr{B}_0$ and $\mathscr{B}_1$ are uniformly relatively $\Delta^0_\alpha$ categorical. Let $\mathscr{A}$ be a $\Delta^0_\alpha$-computable structure such that $CatSpec_{0^{(\alpha)}}(\mathscr{A}) \subseteq \{\mathbf{d} \mid \mathbf{0}^{(\alpha)} \leq \mathbf{d}\}$. Then we can build a structure $\mathscr{N}$ for $\mathscr{A}$ such that

$$CatSpec(\mathscr{N}) = CatSpec_{0^{(\alpha)}}(\mathscr{A}).$$

Using this proposition, we can give a new proof of the following result.

**Theorem 3** ([3], [4]). For every computable ordinal $\alpha$, $0^{(\alpha)}$ is degree of categoricity.

If time permits, we will discuss more applications of the same construction.

### References.


[1] C. J. Ash, J. F. Knight. Pairs of recursive structures. *Ann. Pure Appl. Logic*, 46, 211–234, 1990.

[2] S. V. Vatev. Another jump inversion theorem for structures. *Lecture Notes in Computer Science*, 7921, 414–424, 2013.

[3] E. B. Fokina, I. Kalimullin, and R. Miller. Degrees of categoricity of computable structures. *Arch. Math. Logic*, 49, 51–67, 2010.

[4] B. F. Csima, J. N. Y. Franklin, and R. A. Shore, Degrees of categoricity and the hyperarithmetical hierarchy. *Notre Dame J. of Formal Logic*, 54, 215–232, 2012.


## Entropy and other subshift invariants


Linda Brown Westrick
*University of California, Berkeley*


A subshift is a closed, shift-invariant subset of Cantor space. Also known as symbolic dynamical systems, subshifts were originally used to condense information from continuous dynamical systems. We discuss three subshift invariants: entropy, Medvedev degree, and effective dimension spectrum.

The entropies of $n$-dimensional subshifts of finite type have been characterized by Hochman and Meyerovitch, and the Medvedev degrees of subshifts have been characterized by Miller (in one dimension) and Simpson ($n$-dimensional subshifts of finite type). We extend the existing work on entropy and Medvedev degree to show that these are independent invariants, both in general and for various natural classes of subshifts.

The effective dimension spectrum is defined here as $\{\dim x : x \in X\}$ where $X$ is a subshift. By a result of Simpson, the effective dimension spectrum refines the entropy as an invariant. Seeking a characterization of the effective dimension spectra of subshifts, we discuss motivating examples and partial results.