Coding in the automorphism group of a structure

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Problems

Problem	Instance	Solution
Limit Approximation	$X \in 2^\omega$	$f \in 2^\omega$ s.t. $\lim_s f(\cdot, s) = X$
Domination	$f \in \omega^\omega$	$g \in \omega^\omega$ s.t. $\forall n\, g(n) > f(n)$
Escaping	$f \in \omega^\omega$	$g \in \omega^\omega$ s.t. $\exists^\infty n\, g(n) > f(n)$
Derandomization	$X \in 2^\omega$	$A \ni X$ a null $\mathbf{\Pi}^0_2$-class
Isomorphism	$\mathcal{A}, \mathcal{B}$ structures	$f : \mathcal{A} \cong \mathcal{B}$
Paths	T a tree	$X \in [T]$

Problems

Problems can get you:

- Classes of degrees (highness and lowness notions)
- Weihrauch reducibility
- Dual problems
- Cichoń's diagram
- Reverse math

Most fall into one of two sorts:
First sort – Any countable set of instances share a common solution; any instance computes a solution to itself.
Second sort – Any real X computes an instance (with solutions) such that all solutions compute X.

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Paths	T a tree	$X \in [T]$

Lowness and highness

Two highness notions:

- I compute a simultaneous solution to every computable instance.
- For every computable instance, I compute a solution.

And three lowness notions:

- Every instance I compute has a computable solution.
- Every instance that I compute a solution for has a computable solution.
- Every *computable* instance that I compute a solution for has a computable solution.

Lowness

Problem	Instance	Solution	Lowness
Limit Approx.	$X \in 2^\omega$	$\lim_s f(\cdot, s) = X$	Low (i.e. $A' \equiv_T \emptyset'$)
Dom.	$f \in \omega^\omega$	$\forall n\, g(n) > f(n)$	Hyperimmune free
Escaping	$f \in \omega^\omega$	$\exists^\infty n\, g(n) > f(n)$	Hyperimmune free
Derandom.	$X \in 2^\omega$	null $\mathbf{\Pi}^0_2\, A \ni X$	K-trivial
Iso.	$\mathcal{A}, \mathcal{B}$	$f : \mathcal{A} \cong \mathcal{B}$	Low for Isomorphism
Paths	T	$X \in [T]$	Low for Paths

Those last two rows

Definition

An oracle Y is *low for isomorphism* if any two computable structures with a Y-computable isomorphism have a computable isomorphism.

Definition

An oracle Y is *low for paths for Baire space (for Cantor space)* if any Π^0_1-class in Baire space (in Cantor space) with a Y-computable element has a computable element.

Comparing these problems

The set of isomorphisms between two computable structures can be expressed as a Π^0_1-class:

$$\mathrm{Iso}(\mathcal{A}, \mathcal{B}) = \{(f, f^{-1}) : \mathcal{A} \cong_f \mathcal{B}\}$$

So every computable *Isomorphism*-instance gives a computable *Baire-Paths*-instance, and solutions to one give solutions to the other.

Proposition

Low for Paths for Baire space $\subseteq$ Low for Isomorphism.

Comparing these problems

A Π^0_1-class on Cantor space is a Π^0_1-class on Baire space.

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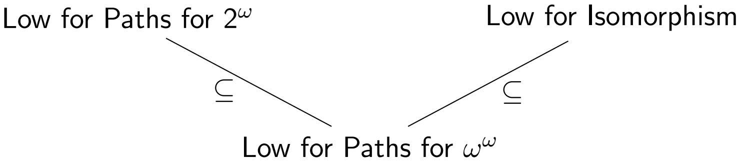

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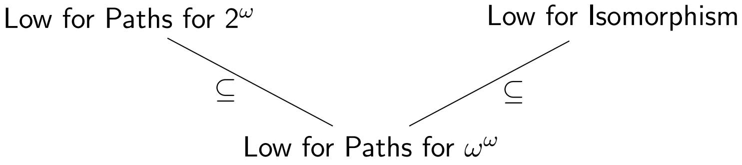


Can we collapse this picture?

Muchnik reducibility

For $U, V \subseteq \omega^\omega$, we say that U is *Muchnik-below* V, written $U \leq_w V$, if every element of V computes an element of U.

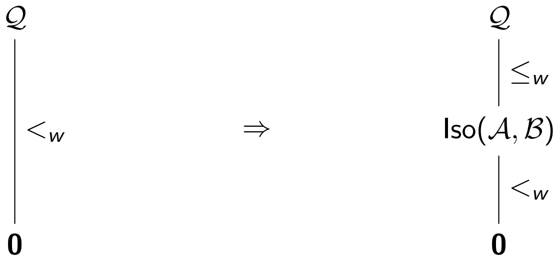
We get a degree notion: $U \equiv_w V$ if $U \leq_w V$ and $V \leq_w U$.

Top element: $\emptyset$.

Bottom degree: $\mathbf{0}$, consisting of all sets containing computable elements.

Restating the question

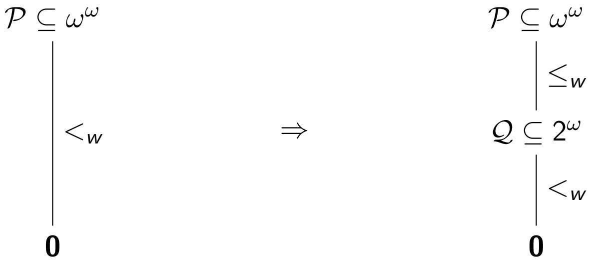
For every Π^0_1-class $\mathcal{Q}$ (from Baire or Cantor space) with $\mathbf{0} <_w \mathcal{Q}$, are there computable structures $\mathcal{A}, \mathcal{B}$ with $\mathbf{0} <_w \text{Iso}(\mathcal{A}, \mathcal{B}) \leq_w \mathcal{Q}$?



Reducing to Cantor space

Theorem (Simpson)

For every Π^0_1-class $\mathcal{P} \subseteq \omega^\omega$ with $0 <_w \mathcal{P}$, there is a Π^0_1-class $\mathcal{Q} \subseteq 2^\omega$ with $0 <_w \mathcal{Q} \leq_w \mathcal{P}$.



Answering the question

Theorem (Franklin & T.)

For every Π^0_1-class $\mathcal{Q} \subseteq 2^\omega$, there is a computable structure $\mathcal{A}$ and elements $a, b \in \mathcal{A}$ such that:

$$\mathcal{Q} \equiv_w Aut(\mathcal{A}) \setminus \{id\} \equiv_w \{f \in Aut(\mathcal{A}) : f(a) = b\}$$

So $\mathcal{Q} \equiv_w \mathrm{Iso}((\mathcal{A}, a), (\mathcal{A}, b))$.

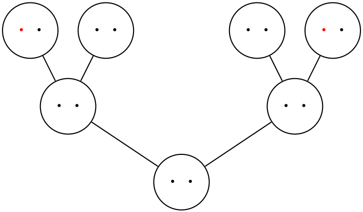
Corollary

Low for paths for Baire space = Low for paths for Cantor space = Low for Isomorphism.

Proving the theorem

Fix a computable tree T with $[T] = \mathcal{Q}$.

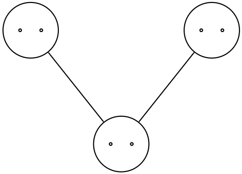
- Domain of $\mathcal{A}$ is $2^{<\omega} \times \{0, 1\}$.
- Unary relations $\{R_\sigma : \sigma \in 2^{<\omega}\}$ with $\mathcal{A} \models R_\sigma((\tau, i))$ iff $\sigma = \tau$.
- Unary relation L with $\mathcal{A} \models L((\sigma, i))$ iff $i = 0$ and $\sigma \notin T$.



Proving the theorem

One more ingredient:

- A ternary relation S with $\mathcal{A} \models S((\sigma_0, i_0), (\sigma_1, i_1), (\sigma_2, i_2))$ iff:
  - $\sigma_1 = \sigma_0{}^\frown 0$;
  - $\sigma_2 = \sigma_0{}^\frown 1$; and
  - $i_0 + i_1 + i_2$ is even.



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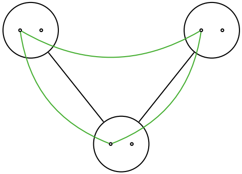

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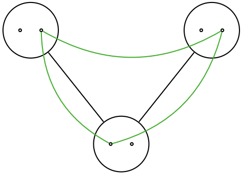

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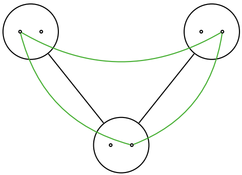

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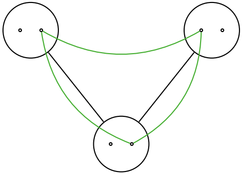
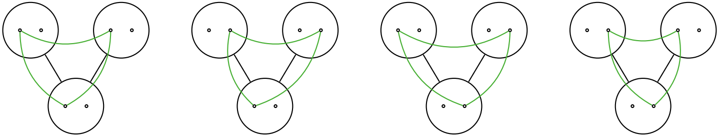


Proving the theorem

- Any automorphism f of $\mathcal{A}$ must respect the R_σ, so $\{f((\sigma,0)), f(\sigma,1)\} = \{(\sigma,0),(\sigma,1)\}$.
- Say f *swaps at* σ if $f((\sigma,0)) = (\sigma,1)$.
- To respect L, f must not swap at any $\sigma \notin T$.
- To respect S, if f must swap at 0 or 2 of σ,$\sigma^\frown 0$, $\sigma^\frown 1$.
- So if f swaps at σ, it must swap at exactly one of $\sigma^\frown 0$, $\sigma^\frown 1$.



Any nontrivial automorphism gives a path via “follow the swaps”.

Proving the theorem

$a = (\langle\rangle, 0)$, $b = (\langle\rangle, 1)$.

Any $X \in \mathcal{Q}$ gives an automorphism: swap precisely at the $\sigma \subset X$.

Since $\langle\rangle \subset X$, this sends a to b. $\square$

Limitations of the result

This is enough for the theorem, but I'd like to do better:

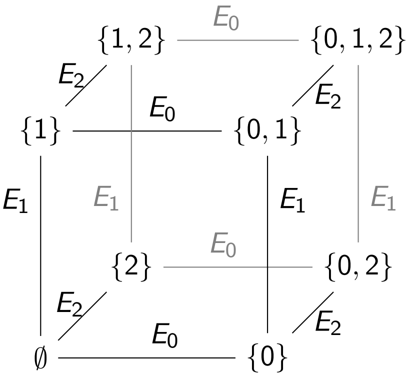
1. Can we handle Π^0_1-classes in Baire space directly, instead of using Simpson's result?
2. Can we code the Π^0_1-class into the isomorphisms between any two computable copies of the structure? Not hard to make a computable $\mathcal{C} \cong (\mathcal{A}, a)$ such that every isomorphism between them computes $\emptyset'$.

Yes and yes.

A new widget

Let $\mathcal{B} = ([\omega]^{<\omega}, (E_i)_{i\in\omega})$, where

$$\mathcal{B} \models E_i(F, G) \leftrightharpoons F \triangle G = \{i\}.$$



$\mathcal{B}$ is the infinite dimensional hypercube with edges colored by direction. Alternatively, it's the affine space $\bigoplus_{i<\omega} \mathbb{Z}/2$.

A new widget

Let $\mathcal{B} = ([\omega]^{<\omega}, (E_i)_{i\in\omega})$, where

$$\mathcal{B} \models E_i(F, G) \leftrightharpoons F \triangle G = \{i\}.$$

The automorphisms of $\mathcal{B}$ are precisely the maps $F \mapsto F \triangle H$ for a fixed H.

So:

- The automorphism group of $\mathcal{B}$ acts transitively (all elements in the same orbit).
- The automorphism group of $\mathcal{B}$ acts freely ($\mathcal{B}$ becomes rigid with the addition of a constant).

Improving to Baire space

Theorem (T)

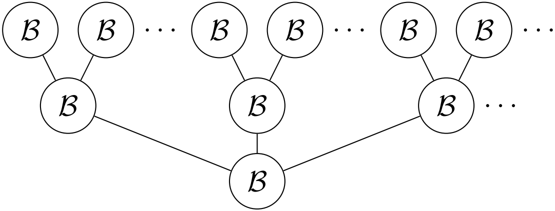
For every Π^0_1-class $\mathcal{Q} \subseteq \omega^\omega$, there is a computable structure $\mathcal{A}$ and element $a \in \mathcal{A}$ such that:

$$\mathcal{Q} \equiv_w Aut(\mathcal{A}) \setminus \{id\} \equiv_w \{f \in Aut(\mathcal{A}) : f(a) \neq a\}$$

Proving the theorem

Fix a computable tree T with $[T] = \mathcal{Q}$.

- Domain of $\mathcal{A}$ is $\omega^{<\omega} \times \mathcal{B}$.
- Unary relations $\{R_\sigma : \sigma \in \omega^{<\omega}\}$ as before.
- Unary relation L with $\mathcal{A} \models L((\sigma, F))$ iff $F = \emptyset$ and $\sigma \notin T$.



Proving the theorem

One more ingredient:

- A binary relation S with $\mathcal{A} \models S((\sigma, F), (\tau, G))$ iff:
  - $\tau = \sigma^\frown i$ for some i; and
  - $i \in F$ iff $|G|$ is odd.

Say that f *moves at* σ if $f((\sigma, \emptyset)) \neq (\sigma, \emptyset)$.

If f is an automorphism with $f((\sigma, \emptyset)) = (\sigma, F)$ and $i \in F$, then f moves at $\sigma^\frown i$.

So nontrivial automorphisms give paths via "follow the movement".

$a = (\langle\rangle, \emptyset)$. Paths give automorphisms moving a via "move along the path". $\square$

What about other copies?

The easy way to control the other computable copies of $\mathcal{A}$ is to arrange that there aren't any.

Definition

A computable structure $\mathcal{A}$ is *computably categorical* if every computable $\mathcal{B}$ with $\mathcal{B} \cong \mathcal{A}$ is computably isomorphic.

So there is only one computable copy, modulo computable isomorphism.

Adding computable categoricity

Theorem (T)

For every Π^0_1-class $\mathcal{Q} \subseteq \omega^\omega$, there is a closed set $\mathcal{R} \subseteq \omega^\omega$, a computable structure $\mathcal{A}$ and an element $a \in \mathcal{A}$ such that:

- *$\mathcal{R}$ is Δ^0_3-homeomorphic to $\mathcal{Q}$;*
- *$\mathcal{A}$ is computably categorical; and*
- $\mathcal{R} \equiv_w Aut(\mathcal{A}) \setminus \{id\} \equiv_w \{f \in Aut(\mathcal{A}) : f(a) \neq a\}$.

As previous theorem, adding a Δ^0_3-priority construction for computable categoricity.

What can we get from this?

Corollary

There is a computably categorical structure of Scott rank $\omega_1^{ck}+1$.

Proof.

Use $\mathcal{Q}$ the Π^0_1-class of descending sequences through the Harrison order. □

Effective dimension

For a computable structure $\mathcal{A}$, the *effective dimension* is the number of computable copies of $\mathcal{A}$, modulo computable isomorphism.

Any natural structure has effective dimension 1 or $\aleph_0$.

Another corollary

Corollary

There is a structure of computable dimension 2 such that the two copies have no Δ^1_1 isomorphism between them.

Previous best known was no Δ^0_2 isomorphism between them.

Proof.

Use $\mathcal{Q} = \{0^\frown X : X$ is a desc. seq. through the Harrison order$\}$.

The two copies are $(\mathcal{A}, (\langle\rangle, \emptyset))$ and $(\mathcal{A}, (\langle\rangle, \{0\}))$. □

Degree spectra

The isomorphism spectrum for a pair of structures is the set of oracles computing an isomorphism:

$$\mathrm{IsoSpec}(\mathcal{A}, \mathcal{B}) = \{Y : \exists f \leq_T Y : \mathcal{A} \cong_f \mathcal{B}\}$$

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The categoricity spectrum for a computable structure is the set of oracles computing an isomorphism between any two computable copies:

$$\mathsf{CatSpec}(\mathcal{A}) = \bigcap_{\substack{\mathcal{B},\mathcal{C} \in \Delta^0_1 \\ \mathcal{B},\mathcal{C} \cong \mathcal{A}}} \mathsf{IsoSpec}(\mathcal{B}, \mathcal{C})$$

Spectral Dimension

$$\mathrm{CatSpec}(\mathcal{A}) = \bigcap_{\substack{\mathcal{B},\mathcal{C}\in\Delta^0_1 \\ \mathcal{B},\mathcal{C}\cong\mathcal{A}}} \mathrm{IsoSpec}(\mathcal{B},\mathcal{C})$$

The spectral dimension of a computable structure is the minimum size of the above intersection:

$$\mathrm{SpecDim}(\mathcal{A}) = \min\{|F| : F \subseteq \Delta^0_1, \mathrm{CatSpec}(\mathcal{A}) = \bigcap_{\substack{\mathcal{B},\mathcal{C}\in F \\ \mathcal{B},\mathcal{C}\cong\mathcal{A}}} \mathrm{IsoSpec}(\mathcal{B},\mathcal{C})\}$$

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Any natural structure has spectral dimension 2.

Question (Kalimullin et al.)

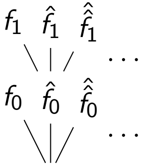
Is there a computable structure $\mathcal{A}$ such that $\mathrm{CatSpec}(\mathcal{A})$ has a least element, and $\mathrm{SpecDim}(\mathcal{A}) = \aleph_0$?

Another way to look at this

Suppose CatSpec($\mathcal{A}$) has a least element X. Let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ be the computable copies of $\mathcal{A}$.

Build a tree of finite sequences of functions, $\langle f_0, \ldots, f_{k-1} \rangle \in (\omega^\omega)^*$ such that:

- If $\langle f_0, \ldots, f_{k-1} \rangle$ is on the tree, and $f_0 \oplus \cdots \oplus f_{k-1} \geq_T X$, then it is a leaf.
- Otherwise, the children of $\langle f_0, \ldots, f_{k-1} \rangle$ are the $g : \mathcal{A}_k \cong \mathcal{A}$.



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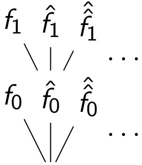


This tree is well-founded (exact pair construction).

Can it have infinite tree-rank?

Answering the previous question

Theorem (T)

There is a computable structure $\mathcal{A}$ *such that* $CatSpec(\mathcal{A}) = \{Y : Y \geq_T \emptyset''\}$ *and* $SpecDim(\mathcal{A}) = \aleph_0$.

Answering the previous question

Theorem (T)

There is a computable structure $\mathcal{A}$ such that $CatSpec(\mathcal{A}) = \{Y : Y \geq_T \emptyset''\}$ and $SpecDim(\mathcal{A}) = \aleph_0$.

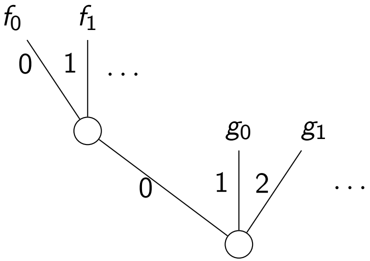
Lemma

There are $f_0, f_1, \ldots, g_0, g_1, \cdots \in \omega^\omega$ s.t.

- *Each $\{f_i\}$ and $\{g_j\}$ is a Π^0_1-class, uniformly;*
- *For any $i < j$, $f_i + g_j \geq_T \emptyset''$; and*
- *For any i, $g_0 \oplus g_1 \oplus \cdots \oplus g_{i-1} \oplus f_i \not\geq_T \emptyset'$.*

Proving the theorem

Use $\mathcal{Q} = \{(j+1)^\frown g_j, 0i^\frown f_i : i, j \in \omega\}$.



Structure is $(\mathcal{A}, (\langle\rangle, \emptyset))$.