A local approach to uniform Martin's conjecture

joint work with Patrick Lutz

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Theorem (Friedberg and Muchnich, independently in 1957 and 1956 resp.)

There exists $i, j \in \omega$ s.t. W_i^x and W_j^x are $\leq_T$-incomparable for all $x \in 2^\omega$. In particular,

$$x <_T W_i^x <_T x'$$

for all $x \in 2^\omega$.

Nevertheless, this doesn't define a canonical non-complete c.e.-in-$[x]_T$ degree, as the map $x \mapsto W_i^x$ doesn't pass to Turing degrees.

Question (Sacks, 1967)

Is there an $e \in \omega$ such that

$$x \equiv_T y \implies W_e^x \equiv_T W_e^y$$
$$x <_T W_e^x <_T x'$$

for all $x, y \in 2^\omega$?

A common way to phrase this question is: 'Is there a *degree invariant* solution to Post's problem?'

Indeed, a function $f : 2^\omega \to 2^\omega$ is called **degree invariant** (**DI**) if $x \equiv_T y \implies f(x) \equiv_T f(y)$.

In some sense, Sacks' question asks whether there is a "natural" solution to Post's problem.

Martin's conjecture aims at generalizing Sacks' question and providing an explanation for the following:

Empirical phenomenon

Turing degrees of naturally occurring problems seem to be well-ordered by $\leq_T$.
Moreover, there seem to be no "natural" degree between a natural degree and its jump.

$$\mathrm{Inf} = \{\, n \mid \text{for infinitely many } i,\ \varphi_n(i) \text{ converges} \,\} \equiv_T 0''$$
$$\mathrm{Cof} = \{\, n \mid \text{for cofinally many } i,\ \varphi_n(i) \text{ converges} \,\} \equiv_T 0'''$$

$$\{\, n \mid \text{the sentence coded by } n \text{ is true in } \mathbb{N} \,\} \equiv_T 0^{(\omega)}$$

$$\mathcal{O} = \{\, n \mid n \text{ is the notation for a computable ordinal} \,\}$$

$$0^\sharp = \{\, n \mid n \text{ codes a true statement about } L \,\}$$

$$\mathrm{Inf}^x = \{\, n \mid \text{for infinitely many } i,\ \varphi_n^x(i) \text{ terminates} \,\} \equiv_T x''$$
$$\mathrm{Cof}^x = \{\, n \mid \text{for cofinally many } i,\ \varphi_n^x(i) \text{ terminates} \,\} \equiv_T x'''$$

For $x \in 2^\omega$, let $\mathbb{N}^x$ be the usual structure $(\mathbb{N}, +, \cdot)$ with an extra unary predicate to be interpreted as x.

$$\{\, n \mid \text{the sentence coded by } n \text{ is true in } \mathbb{N}^x \,\} \equiv_T x^{(\omega)}$$

$$\mathcal{O}^x = \{\, n \mid n \text{ is the notation for a computable-in-}x \text{ ordinal} \,\}$$

$$x^\sharp = \{\, n \mid n \text{ codes a true statement about } L[x] \,\}$$

Formalization of "natural" degrees

Idea: any "natural" Turing degree has a definition that can be relativized to arbitrary $x \in 2^\omega$, leading to a *definable* degree invariant function $f : 2^\omega \to 2^\omega$.

So maybe the structure of *non-trivial*, *definable* DI functions is *essentially* well-ordered.

In Martin's conjecture, "definable" functions = arbitrary functions under AD.

"Essentially" = up to Martin's measure.

Definition

Turing determinacy (TD) is the statement that every Turing invariant (i.e. closed under $\equiv_T$) $A \subseteq 2^\omega$ either contains a cone

$$\{\, x \in 2^\omega \mid x \geq_T z \,\}$$

or is disjoint from a cone.

Theorem (Martin)

AD implies TD.

Under TD,

$$\mu(A) = \begin{cases} 1 & \text{if } A \text{ contains a cone} \\ 0 & \text{if } 2^\omega \setminus A \text{ contains a cone} \end{cases}$$

is a measure on the family of Turing invariant subset of 2^ω called Martin's measure.

Given DI functions $f, g : 2^\omega \to 2^\omega$, define

$$f \leq_M g \iff f(x) \leq_T g(x) \text{ for almost every } x$$
$$f \equiv_M g \iff f(x) \equiv_T g(x) \text{ for almost every } x.$$

The intuitive idea behind Martin's conjecture is that "natural" Turing degrees induce "natural" DI functions (by relativizing their definition).
Moreover, under AD, we don't have many ways to cook up non-"natural" DI functions:
we can patch together $\aleph_0$ many natural DI functions into a single one... but by TD one of the pieces prevails almost everywhere!

So if we take the set of non-trivial DI functions under AD and we mod out by $\equiv_M$, we *morally* obtain the set "natural" Turing degrees.

But we need to exclude functions that are constant on a cone, because $\leq_M$ on them up to $\equiv_M$, corresponds to $\leq_T$ on Turing degrees: these are the *trivial* functions.

Conjecture (Martin)

Under ZF+DC+AD:

1. if $f : 2^\omega \to 2^\omega$ is DI, then either f is constant (up to $\equiv_T$) on a cone, or $f(x) \geq_T x$ on a cone;
2. the set

$$\{\, f : 2^\omega \to 2^\omega \mid f \text{ is DI and } f(x) \geq_T x \text{ on a cone} \,\}$$

   is pre-wellordered by $\leq_M$.
   Moreover, if f has rank α, f' has rank $\alpha + 1$.

In particular, part II of Martin's conjecture would imply a negative answer for Sacks' question.
Indeed, not only DI functions of the form $x \mapsto W_e^x$, but all DI functions f whose existence can be proved in ZF+DC+AD could not satisfy

$$x <_T f(x) <_T x'$$

for almost all x.

Uniformly invariant functions

Say that $x \leq_T y$ via i if $\varphi_i^y = x$.

Say that $x \equiv_T y$ via (i, j) if $x \geq_T y$ via i and $x \leq_T y$ via j.

Let $A \subseteq 2^\omega$. We say that $f : A \to 2^\omega$ is **uniformly degree invariant (UDI)** if there is $u : \omega^2 \to \omega^2$ such that

$$x \equiv_T y \text{ via } (i, j) \implies f(x) \equiv_T f(y) \text{ via } u(i, j).$$

Such a u is called **uniformity function** for f.

Note that (f, u) can be viewed as a homomorphism of the two-sorted relation '$\equiv_T$ via'.

Theorem (Lachlan, 1975)

There exists no *uniformly* degree invariant solution to Post's problem.
In other words, there are no $e \in \omega$ and $u : \omega^2 \to \omega^2$ such that

$$x \equiv_T y \text{ via } (i, j) \implies W_e^x \equiv_T W_e^y \text{ via } u(i, j)$$
$$x <_T W_e^x <_T x'$$

for all $x, y \in 2^\omega$.

Theorem (Steel, 1983)

Part II of Martin's conjecture holds for UDI functions.

Theorem (Slaman-Steel, 1988)

Part I of Martin's conjecture holds for UDI functions.

Theorem (Becker, 1988)

(ZF+DC+AD) For every UDI function f such that $f(x) >_T x$ on a cone, there exist a lightface pointclass Γ such that

$$f(x) \equiv_T \text{universal } \Gamma(x) \text{ subset of } \omega.$$

Theorem (Kihara-Montalbán, 2016)

(ZF+DC+AD) There is an isomorphism between the partial ordering of $\equiv_m^\triangledown$-degrees of Turing- to many-one *uniformly* invariant functions ordered by

$$f \leq_m^\triangledown g \iff \exists z \in \omega^\omega : \forall x \geq_T z : f(x) \leq_m^z g(x)$$

and the partial ordering of Wadge degrees of subsets of ω^ω ordered by Wadge reducibility.

Conjecture (Steel)

Under ZF+DC+AD, every DI function is $\equiv_M$ to a UDI-on-a-cone one.

This would instantly give us a lot of knowledge on DI functions.

In particular, as a consequence of Slaman and Steel's results, this conjecture implies Martin's.

The local approach

Notice that, while the structure of $\equiv_T$ on each $[x]_{\equiv_T}$ is trivial, the structure of "$\equiv_T$ via" on each $[x]_{\equiv_T}$ is *not* trivial.

We are going to explore the relation between the structure of "$\equiv_T$ via" on single degrees and uniform Martin's conjecture.

For instance: what are the possible degrees of the range of a UDI function $f : [x]_T \to 2^\omega$?

It can be any $[y]_T \geq_T [x]_T$: for example consider $z \mapsto z \oplus y$. Are there other possibilities?

Main results

Theorem (B.)

If $f : [x]_T \to 2^\omega$ is UDI, then either f is constant or $f(x) \geq_T x$.

Theorem (Slaman and Steel, 1988)

(ZF+DC+AD) If $f : 2^\omega \to 2^\omega$ is UDI on a cone, then either f is constant on a cone, or $f(x) \geq_T x$ on a cone.

Theorem

If $f : [x]_T \to 2^\omega$ is UDI, then either f is constant or $f(x) \geq_T x$.

Corollary (Slaman and Steel's 1988 result, improved version)

(ZF+DC+TD) For every UDI $f : 2^\omega \to 2^\omega$, either f is constant on a cone, or $f(x) \geq_T x$ on a cone.

Proof of the Corollary given the Theorem.

By TD, there is a cone C s.t. either $f \restriction [x]_T$ is constant for all $x \in C$ or $f \restriction [x]_T$ is non-constant for all $x \in C$.
In the latter case, $f(x) \geq_T x$ on C, by the Theorem.
In the former case, on C we have

$$x \equiv_T y \implies f(x) = f(y).$$

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$$x \equiv_T y \implies f(x) = f(y).$$

Thus the sets $\{\, x \in C \mid f(x)(k) = 1 \,\}$ are closed under $\equiv_T$, so again by TD, $f(x)(k)$ is constant on a cone C_k, so $f(x)$ is literally constant on $\bigcap_k C_k$, which trivially contains a cone. $\square$

Theorem

The following are equivalent over ZF+DC:

1. TD
2. part I of uniform Martin's conjecture

Proof.

We've just seen 1 $\implies$ 2.
2 $\implies$ 1: if A is Turing invariant, define

$$f(x) = \begin{cases} \underline{0} = 000\ldots & \text{if } x \in A, \\ \underline{0}' & \text{if } x \in 2^\omega \setminus A. \end{cases}$$

f is UDI and it cannot be that $f(x) \geq_T x$ on a cone.
So either $f(x) = \underline{0}$ on a cone, or $f(x) = \underline{0}'$. In the former case, A contains a cone, in the latter $2^\omega \setminus A$ does. $\square$

Theorem (Lachlan, 1975)

There is no UDI map $x \mapsto W_e^x$ such that

$$x <_T W_e^x <_T x'$$

for all $x \in 2^\omega$.

can be derived by the following "local" result:

Theorem (B. and Lutz)

If $z \mapsto W_e^z$ is UDI on $[x]_T$ and $x \geq_T \underline{0}'$, then

- either $W_e^x \equiv_T x'$
- or $W_e^x \equiv_T x$
- or W_e^z is constantly equal to some W_a.

Plan

- proving the two local theorems
- discussing if part II of uniform Martin's conjecture arises locally
- open problems
- an application to computable reducibility, if time suffices

Proof of the main theorems

Lemma

If $f : A \to 2^\omega$ is UDI and A is Turing-invariant (i.e. closed under $\equiv_T$), then there is a *computable* uniformity function for f.

Proof.

Let's prove the easier case in which f is uniformly *order-preserving*, first. This means there is $u : \omega \to \omega$ s.t.

$$x \geq_T y \text{ via } i \implies f(x) \geq_T f(y) \text{ via } u(i).$$

Denote by ij the program that composes program i with program j, that is

$$\varphi_i^{\varphi_j^x} = \varphi_{ij}^x.$$

Note that operation is computable and associative.

Let $a, b, c \in \omega$ s.t.

$$\varphi_c^x = 1^\frown x \qquad \varphi_b^x = 0^\frown x \qquad \varphi_a^{0^e 1^\frown x} = \varphi_e^x.$$

Now, fix $x \in A$ and $e \in \omega$ such that φ_e^x is in A and notice that we have $\varphi_e^x = \varphi_{ab^e c}^x$.
Therefore,

$$f(\varphi_e^x) = f\left(\varphi_{ab^e c}^x\right) \leq_T f\left(\varphi_{b^e c}^x\right) \text{ via } u(a),$$

since $\varphi_{ab^e c}^x \leq_T \varphi_{b^e c}^x$ via a and $\varphi_{b^e c}^x = 0^e 1^\frown x \equiv_T x \in A$.
By iterating this, we finally reach

$$f(\varphi_e^x) \leq_T f(x) \text{ via } u(a)u(b)^e u(c).$$

So define $v(e) = u(a)u(b)^e u(c)$
and you'll have

- $f(\varphi_e^x) \leq_T f(x)$ via $v(e)$, whenever φ_e^x is in A.
  That is, v is a uniformity function for f;
- v is computable.

If f is UDI and v is a uniformity function for it, let:

$$\varphi_c^x = 1^\frown x \qquad \varphi_b^x = 0^\frown x \qquad \varphi_m^x = x^-.$$

We have $x \equiv_T 1^\frown x$ via (c, m) and $x \equiv_T 0^\frown x$ via (b, m).

Let $d \in \omega$ be such that

$$\varphi_d^{0^i 1 0^j 1^\frown x} = 0^j 1 0^i 1^\frown \varphi_i^x.$$

$$\begin{aligned} x \equiv_T y \text{ via } (i, j) &\iff (0^i 1 0^j 1^\frown x) \equiv_T (0^j 1 0^i 1^\frown y) \text{ via } (d, d) \\ &\iff x \equiv_T (0^j 1 0^i 1^\frown y) \text{ via } (d, d)(b, m)^i (b, c)(b, m)^j (c, m) \\ &\iff x \equiv_T y \text{ via } (m, c)(m, b)^i (m, c)(m, b)^j (d, d)(b, m)^i (b, c)(b, m)^j (c, m) \end{aligned}$$

So define $v(i, j)$ as

$u(m, c)u(m, b)^i u(m, c)u(m, b)^j u(d, d)u(b, m)^i u(b, c)u(b, m)^j u(c, m)$.

Now we finally prove

Theorem

If $f : [x]_T \to 2^\omega$ is UDI, then either $f(x) \geq_T x$ or f is constant.

Proof.

We suppose we have $y \in [x]_T$ s.t. $f(x) \neq f(y)$ and we prove $f(x) \geq_T x$.

WLOG, suppose $f(x)(k) = 1$ and $f(y)(k) = 0$.
Define

$$x_n = \begin{cases} x & x(n) = 1 \\ y & x(n) = 0 \end{cases}$$

and notice that $f(x_n)(k) = x(n)$ and that there is a computable $t : \omega \to \omega$ and $e \in \omega$ such that

$$x \equiv_T x_n \text{ via } (t(n), e).$$

If u is a computable uniformity function for f, we have

$$f(x) \equiv_T f(x_n) \text{ via } u(t(n), e).$$

Then, if π is the projection the first coordinate,

$$\varphi^{f(x)}_{\pi u(t(n),e)} = f(x_n).$$

So x is exactly $n \mapsto \varphi^{f(x)}_{\pi u(t(n),e)}(k)$, which is computable in $f(x)$. $\square$

Theorem (joint with Patrick Lutz)

Suppose that $x \geq_T 0'$ and the c.e. operator

$$W_e : z \mapsto W_e^z$$

is uniformly Turing invariant on $[x]_T$. Then, if W_e is discontinuous in x, $W_e^x \equiv_T x'$. Otherwise, if it's continuous in x, either W_e^z is constantly equal to a c.e. set for all $z \in [x]_T$, or $W_e^x \equiv_T x$.

Proof.

First, note that, for all $m \in \omega$, we have $W_e^x(m) = 1$ iff there is a finite initial segment σ of x s.t. $W_e^\sigma(m) = 1$, and hence $W_e^z(m) = 1$ for all z extending σ. Thus, W_e is continuous in x iff

$$\forall m \Big(W_e^x(m) = 0 \iff \exists \sigma \prec x \forall \tau \big(W_e^{\sigma^\frown \tau}(m) = 0 \big) \Big). \tag{1}$$

Suppose that W_e is continuous in x. Since the property $\forall \tau \big(W_e^{\sigma^\frown \tau}(m) = 0 \big)$ is decidable in $0' \leq_T x$, we have that W_e^x is co-c.e. in x. Of course, W_e^x is also c.e. in x, hence $W_e^x \leq_T x$.

Since $z \mapsto W_e^z$ is uniformly degree invariant on $[x]_T$, we know that it's either constant on $[x]_T$ or $W_e^x \geq_T x$.
In the latter case, $W_e^x \equiv_T x$. In the former one, $W_e^x(m) = 1$ iff there is some $z \equiv_T x$ and some initial segment σ of z such that $W_e^\sigma(m) = 1$. Thus, since every Turing degree is dense in 2^ω, we have

$$W_e^x(m) = 1 \iff \exists \sigma \in 2^\omega \big(W_e^\sigma(m) = 1\big)$$

so $[x]_T \ni z \mapsto W_e^z$ is constantly equal to the c.e. set

$$\{\, m \in \omega \mid \exists \sigma \in 2^{<\omega} : W_e^\sigma(m) = 1 \,\}.$$

Now, let's suppose that W_e is *not* continuous in x. So in this case, we must have

$$\exists m \in \omega\big(W_e^x(m) = 0 \wedge \forall \sigma \prec x \exists \tau \in 2^{<\omega} : W_e^{\sigma^\frown \tau}(m) = 1\big).$$

Note that, there's in fact an effective procedure that, given such a σ, finds a suitable τ. Let's denote τ_σ the τ found in this way. So, if we denote by $x'[l]$ the l-th approximation of the jump of x, we can define a computable r such that

$$\varphi_{r(n)}^x = \begin{cases} x & n \notin x' \\ (x \restriction l)^\frown \tau_{x \restriction l}{}^\frown x & \text{if } l \text{ is the least such that } n \in x'[l]. \end{cases}$$

Let's call $\varphi_{r(n)}^x = x_n$. There's also a computable s such that $\varphi_{s(n)}^{x_n} = x$ for all n. Thus, we have $x \equiv_T x_n$ via $(r(n), s(n))$ with r and s computable and $W_e^{x_n}(m) = 1$ iff $x'(n) = 1$. So the argument of our other main theorem gives us $W_e^x \geq_T x'$. □

Open questions

Does the *second* part of uniform Martin's conjecture (that is, Steel's theorem) arise locally, too?

Steel's theorem for Borel functions

Fix $\alpha < \omega_1$. Then, if $f : 2^\omega \to 2^\omega$ Borel UDI function s.t. $f(x) \geq_T x$ on a cone, either there is $\beta < \alpha$ s.t. $f(x) \leq_T x^{(\beta)}$ on a cone, or $f(x) \geq_T x^{(\alpha)}$ on a cone.

Question

Fix $\alpha < \omega_1$ and $x \in 2^\omega$ s.t. $\alpha < \omega_1^x$. Is it true that, given a Borel UDI $f : [x]_T \to 2^\omega$, either f is Baire class β for some $\beta < \alpha$ or $f(x) \geq_T x^{(\alpha)}$?

The function that maps x to the theory of the two-sorted structure $[x]_T$ with '$\equiv_T$ via' is a Borel map $t : 2^\omega \to 2^\omega$ s.t.

$$x \equiv_T y \implies t(x) = t(y),$$

so it has to be constant on a cone.

Question

What is the "almost certain" theory of single Turing degrees? What is its Turing degree? What Turing degrees have it?

Remark

The sentence '$\exists i \, \exists x \, \forall y : x \leq_T y$ via i' is true when interpreted in $[0]_T$ and false when interpreted in all other degrees. So not all Turing degrees are elementarily equivalent wrt the language of '$\leq_T$ via'.

Are all Turing degrees elementarily equivalent wtr the language of '$\equiv_T$ via'?

An application to computable reducibility

Given E and F equivalence relations on ω, we write $E \leq_c F$ when there is a computable $r : \omega \to \omega$ s.t.

$$m \, E \, n \iff r(m) \, F \, r(n).$$

Given $x \in 2^\omega$, define

$$i =^{T,x} j \iff \varphi_i^x = \varphi_j^x.$$

Theorem

The map $x \mapsto =^{T,x}$ is a Borel reduction from $\leq_T$ to $\leq_c$, i.e.

$$x \leq_T y \iff (=^{T,x}) \leq_c (=^{T,y})$$

Proof.

"$\implies$" is easy. Vice versa, if v is a computable reduction from $=^{T,x}$ to $=^{T,y}$, then define

$$f : \{ \varphi_i^x \mid i \in \omega \} \to \{ \varphi_i^y \mid i \in \omega \}$$

by $f(\varphi_i^x) = \varphi_{v(i)}^y$, and notice that f is injective and

$$x \geq_T z \text{ via } i \implies y \geq_T f(z) \text{ via } v(i).$$

So, taking any $z \in \mathrm{dom}(f)$, $z \neq x$, we'll have $f(z) \neq f(x)$, so we can define

$$x_n = \begin{cases} x & n \in x \oplus \bar{x} \\ z & \text{otherwise} \end{cases}$$

and arguing as before we get $y \geq_T \big(n \mapsto f(x_n)(k)\big)$ and thus $y \geq_T x$. $\square$

So $\leq_c$ is at least as complicated as $\leq_T$.

Theorem (Sacks, 1961)

Every locally countable pre-order of cardinality $\leq \aleph_1$ is embeddable into $(2^\omega, \leq_T)$.

Corollary

Every locally countable pre-order of cardinality $\leq \aleph_1$ embeds into $(\mathrm{ER}, \leq_c)$, where ER denotes the set of eq. rel. on ω.

We have $(\leq_T) \leq_B (\leq_c)$. From recent work by Patrick Lutz and Benny Siskind, under analytic determinacy, $\leq_T$ is not a universal locally countable Borel quasi-order.

Question

Is $\leq_c$ a universal locally countable Borel quasi-order?

That's all! Thank you!