A Survey on Analog Models of Computation

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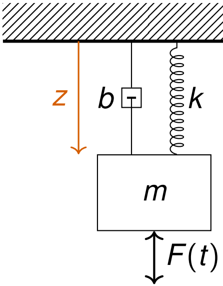
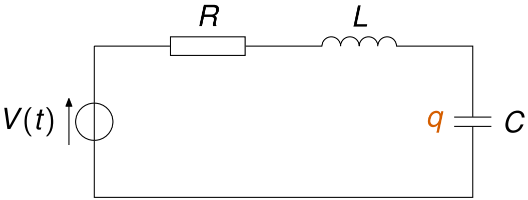
30 june 2020

Survey: `https://arxiv.org/abs/1805.05729`

The meaning of “analog”

Historically: “analog” = by analogy, *i.e.* same evolution





$$F = m\ddot{z} + b\dot{z} + kz \;\Leftrightarrow\; V = L\ddot{q} + R\dot{q} + \frac{1}{C}q$$

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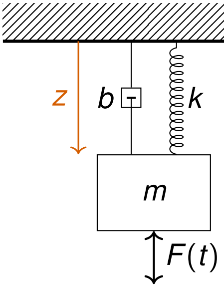

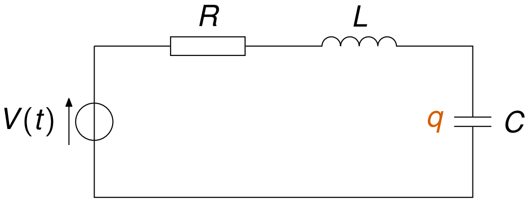

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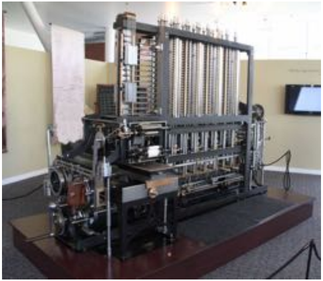
Nowadays: “analog” = continuous/opposite of digital

$\Rightarrow$ orthogonal concepts

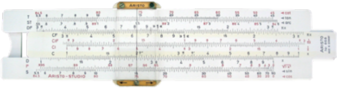
$\Rightarrow$ even continuous/discrete unclear: hybrid exists

Some analog machines



Difference Engine

Linear Planimeter



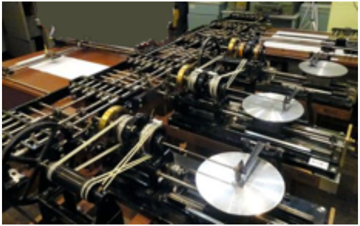
Slide Rule

Antikythera mechanism

Some analog machines

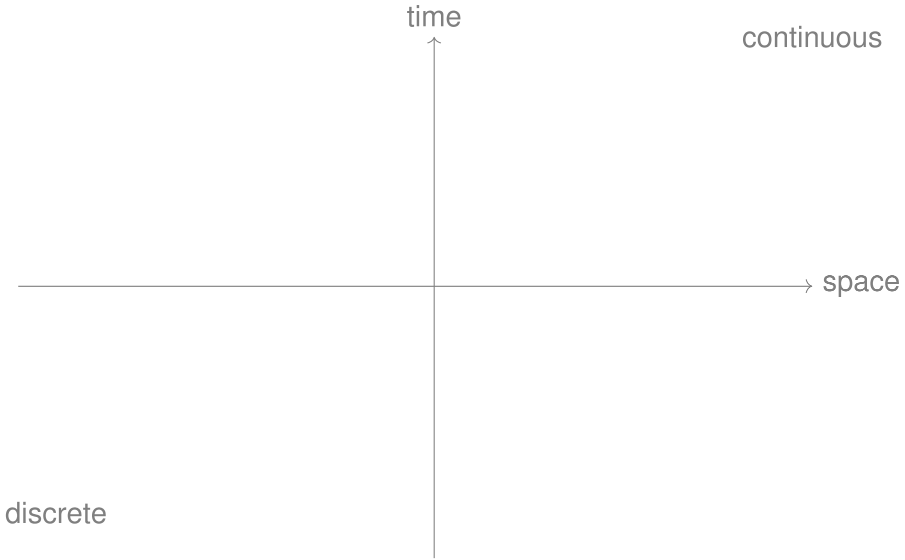

ENIAC


Admiralty Fire Control Table


Differential Analyzer


Kelvin's Tide Predicter

Classifying machines/models



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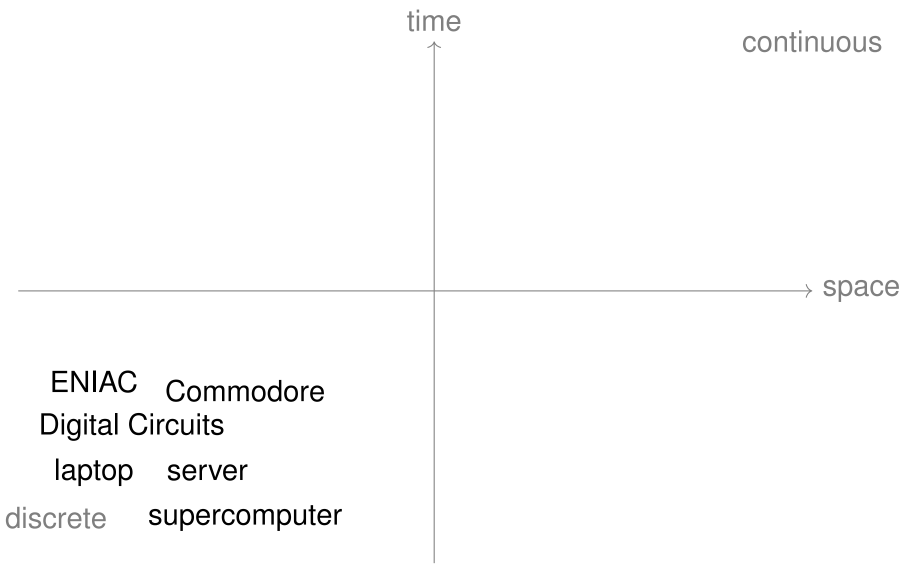

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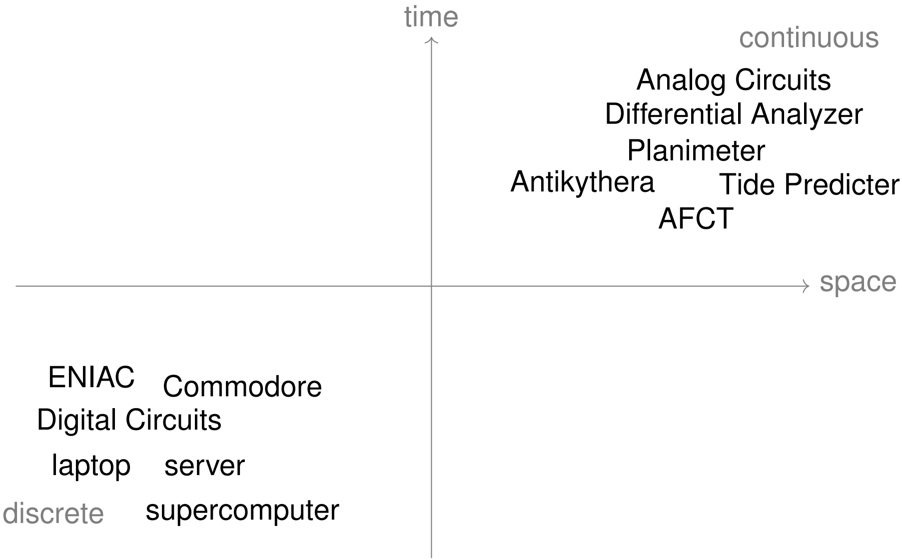

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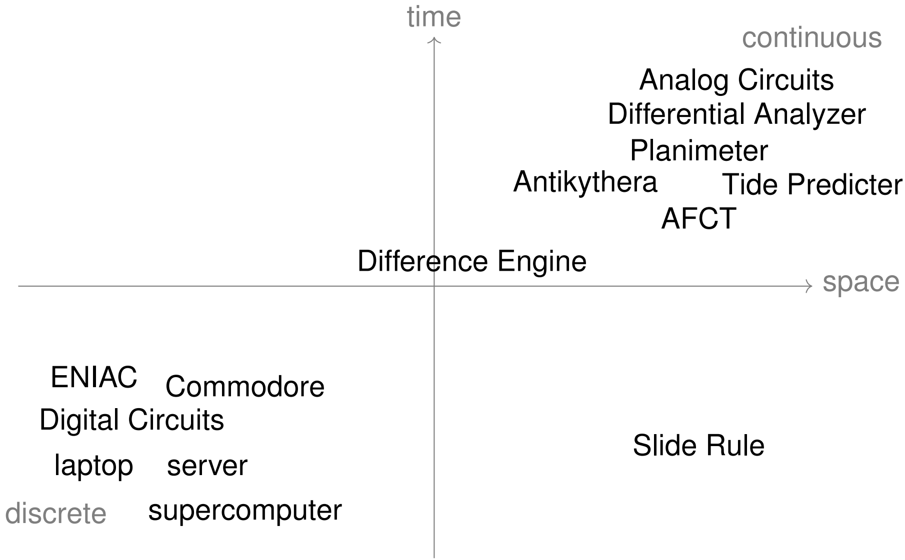

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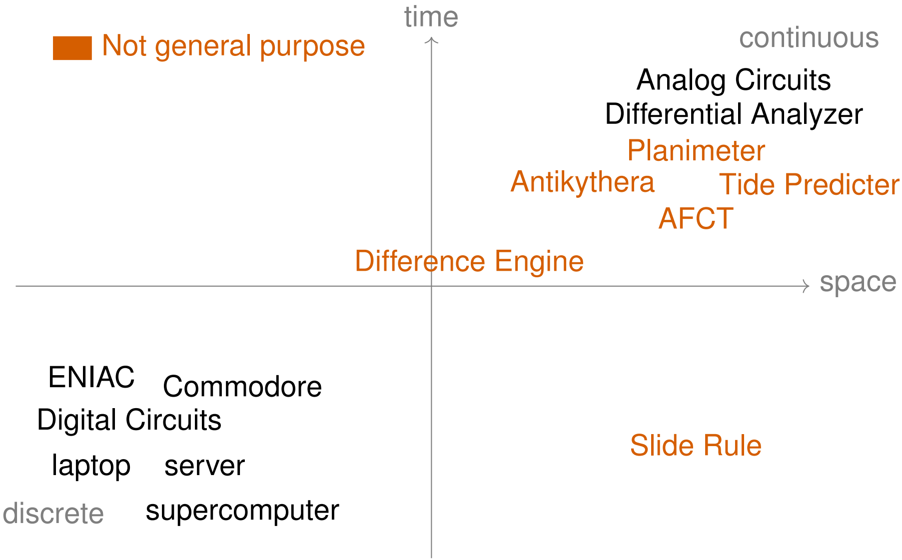

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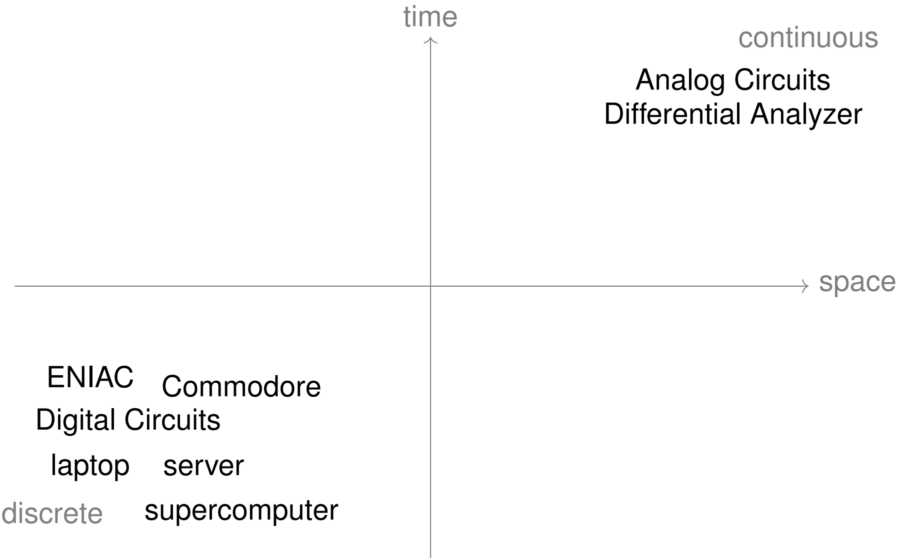


Classifying machines/models

time

Mathematical model

continuous

Analog Circuits
Differential Analyzer

Continuous Dynamical System $y' = f(y)$

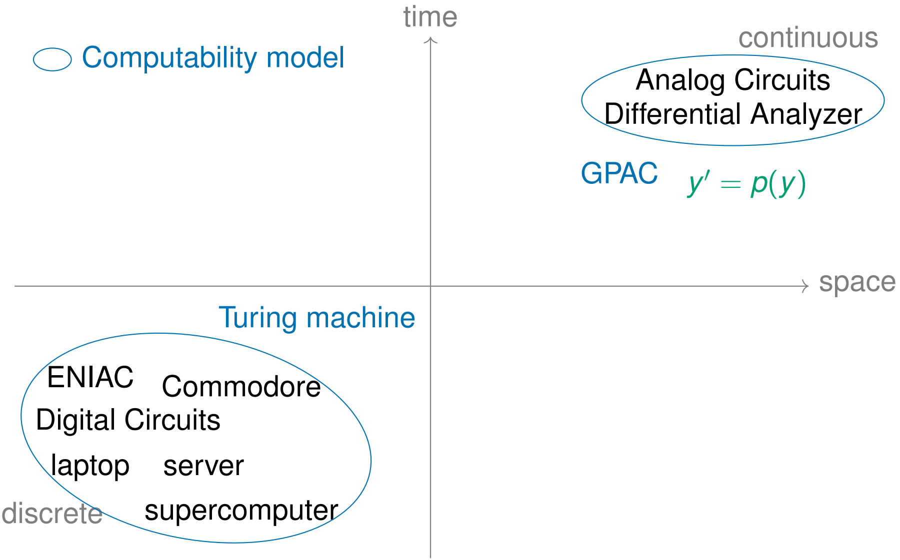
space

Discrete Dynamical System $y_{n+1} = f(y_n)$

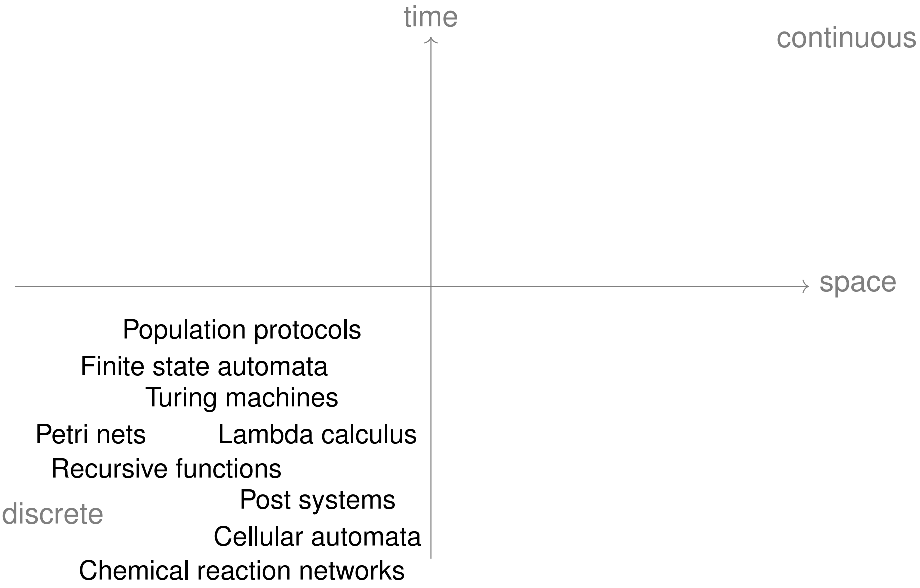
ENIAC
Commodore
Digital Circuits
laptop
server
supercomputer

discrete

Classifying machines/models



The many many models



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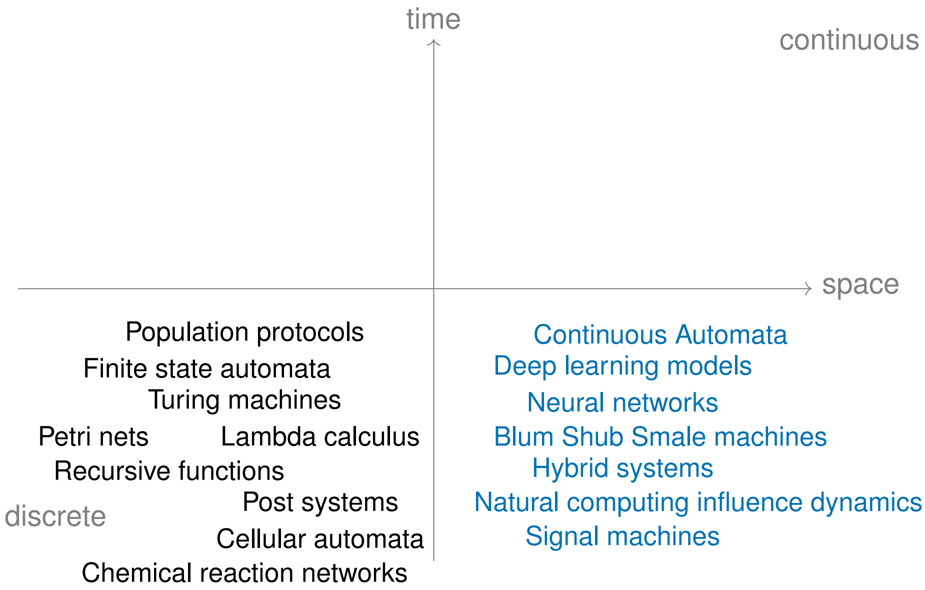

The many many models

time

continuous

$\mathbb{R}-$recursive functions

Large population protocols

Hybrid Systems

Timed automata

Physarum computing

Shannon's GPAC

Black hole models

Hopfield's neural networks

Reaction-Diffusion Systems

Chemical reaction networks

space

Population protocols

Finite state automata

Turing machines

Petri nets

Lambda calculus

Recursive functions

Post systems

Cellular automata

Chemical reaction networks

discrete

Continuous Automata

Deep learning models

Neural networks

Blum Shub Smale machines

Hybrid systems

Natural computing influence dynamics

Signal machines

The many many models

time

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Boolean difference equation models

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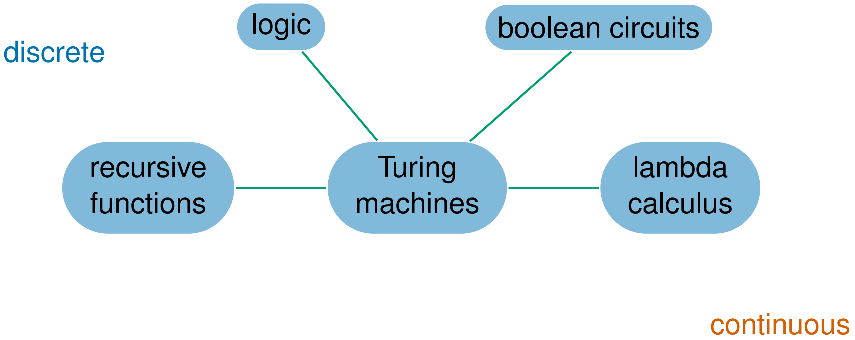
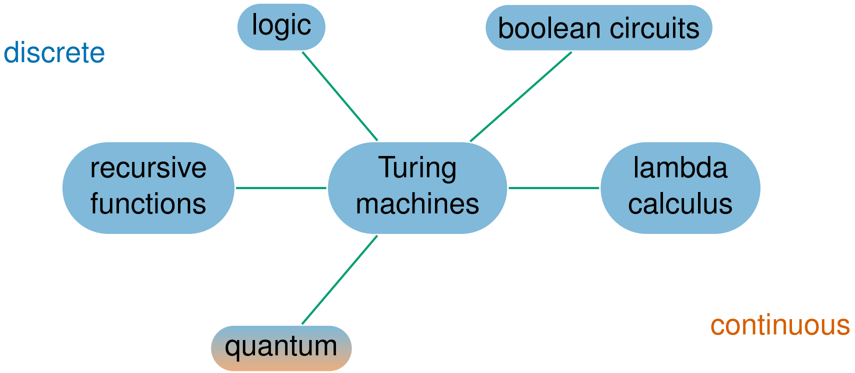
Making sense of all these models



“Church” thesis

All discrete models are Turing machine-computable.

Making sense of all these models



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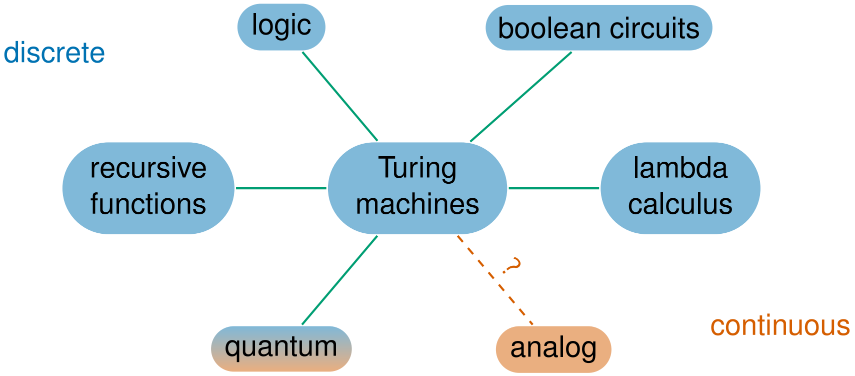
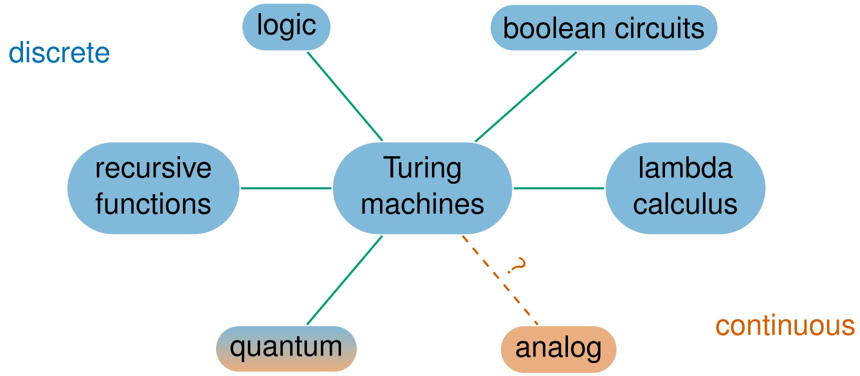


“Church” thesis ?

All models are Turing machine-computable.

Clearly **wrong**: a single real number (Ω of Chaitin) is super-Turing powerful.

Making sense of all these models



"Church" thesis ?

All **physical machine-based** models are Turing machine-computable.

Several issues with that statement.

Machine vs mathematical model

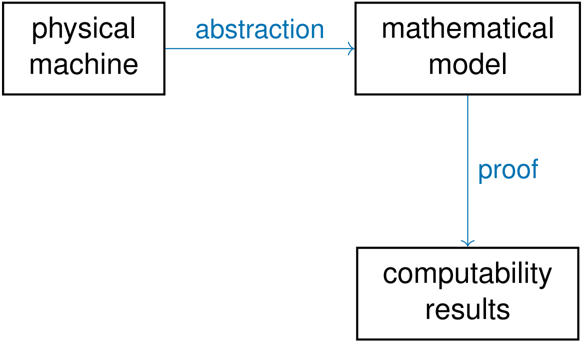
physical
machine

Machine vs mathematical model

$$\text{physical machine} \xrightarrow{\text{abstraction}} \text{mathematical model}$$

- mathematical model = abstraction of a system

Machine vs mathematical model



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- properties of model $\neq$ properties of system

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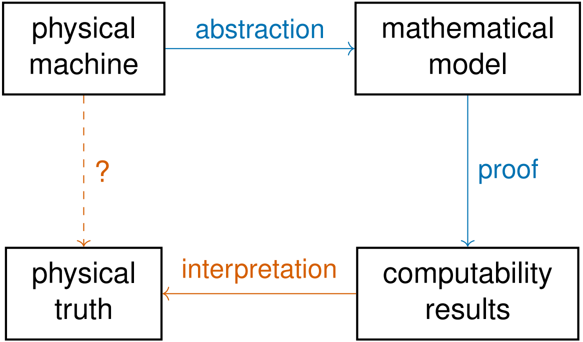
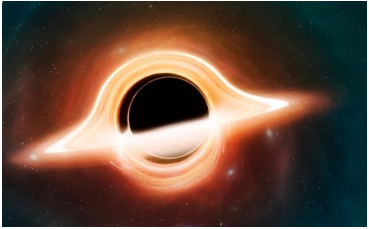


- mathematical model $=$ abstraction of a system
- properties of model $\neq$ properties of system
- conclusion might be quantitatively or qualitatively wrong

Black hole model and hypercomputations

- machine: the universe
- model: general relativity



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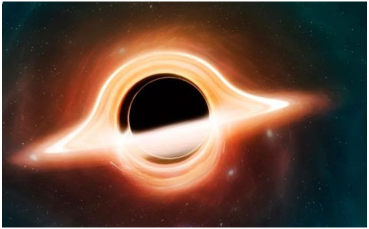

Informal theorem

If slowly rotating Kerr black holes exists, one can check consistency of ZFC or solve the Turing halting problem in finite time.

- conclusion: hypercomputations are possible ?

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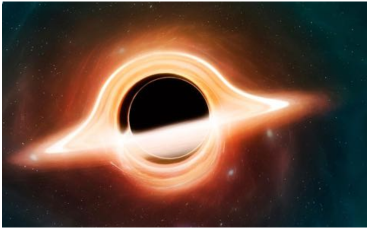

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Common occurrence in analog models: non-computable reals, Zeno phenomena, ...

Back to the Church thesis

Distinguish machines from models:

Actual Church thesis

Every effective computation can be carried out by a Turing machine, and vice versa.

$\Rightarrow$ effective = systematic method in logic/mathematics/CS

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Physical Church Turing thesis/Thesis M

Whatever can be calculated by a machine (with finite data/instructions) is Turing machine-computable.

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Alternative thesis

All **reasonable** models of computations are equivalent to Turing machines.

Chemical Reaction Networks (CRNs)

A **reaction system** is a finite set of

- molecular species $y_1, \ldots, y_n$
- reactions of the form $\sum_i a_i y_i \xrightarrow{f} \sum_i b_i y_i \qquad (a_i, b_i \in \mathbb{N}$, $f =$ rate)

Example (any resemblance to chemistry is purely coincidental):

$$\begin{array}{ccccc} 2H & + & O & \to & H_2O \\ C & + & O_2 & \to & CO_2 \end{array}$$

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Semantics (assuming law of mass action):

- discrete
- differential
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- discrete $\rightarrow$ $\quad$ $y_i =$ molecule count
  close to population protocols
- differential
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- differential $\rightarrow$
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$y_i =$ concentration

polynomial ODEs

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Semantics (assuming law of mass action):

- discrete
- differential
- stochastic $\to$

$y_i =$ probability distribution

stochastic ODEs

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Observation

A system/machine can have several models, all useful, depending on the level of abstraction.

Are analog systems capable of hypercomputations?

Examples: Black holes, signal machines, hybrid systems

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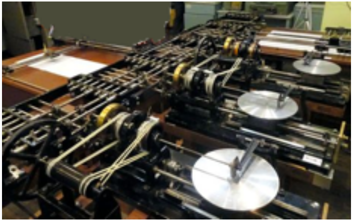
Possible conclusion

All **reasonable** models of computations are equivalent to Turing machines. Hypercomputability results can help us correct models.

General Purpose Analog Computer (GPAC)



Differential analyzer

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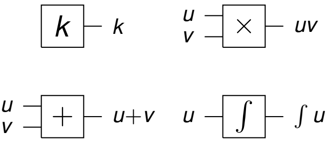
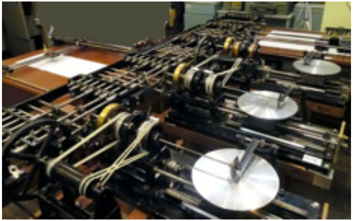


General Purpose Analog Computer, Shannon 1936



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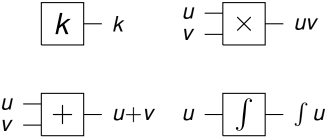

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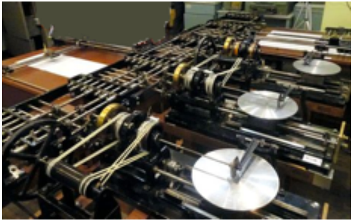
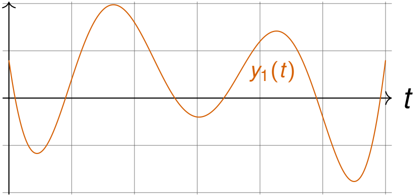


Differential analyzer

$$y(0) = y_0, \quad y'(t) = p(y(t))$$

Polynomial Differential Equation, Graça 2004



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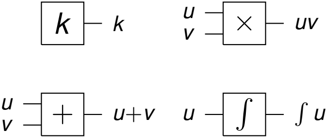

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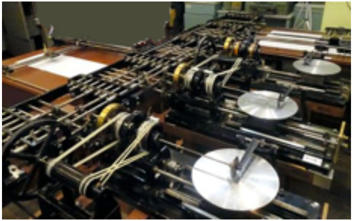

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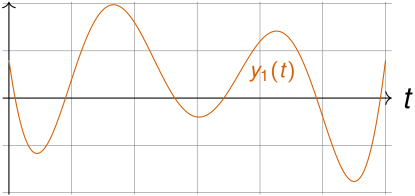
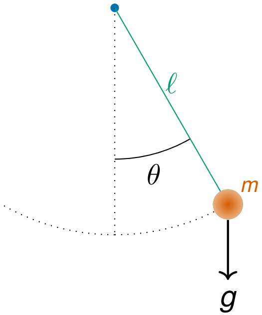


Example of dynamical system



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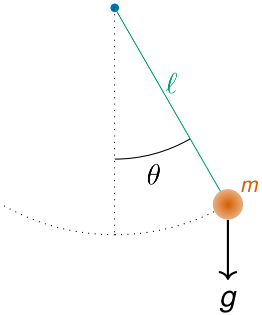

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$$\begin{cases} y_1' = y_2 \\ y_2' = -\frac{g}{l} y_3 \\ y_3' = y_2 y_4 \\ y_4' = -y_2 y_3 \end{cases} \Leftrightarrow \begin{cases} y_1 = \theta \\ y_2 = \dot{\theta} \\ y_3 = \sin(\theta) \\ y_4 = \cos(\theta) \end{cases}$$

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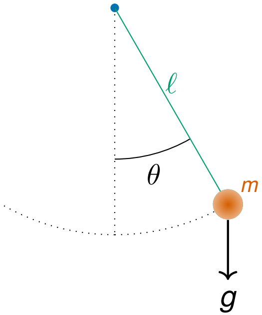

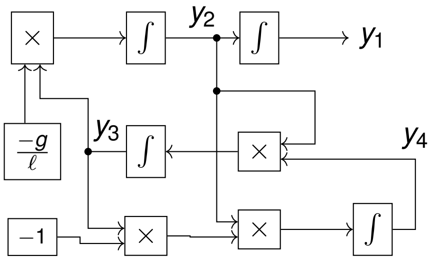

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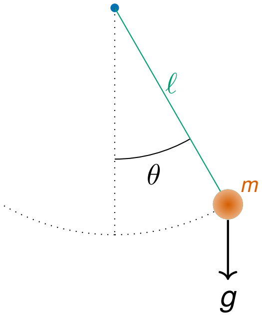

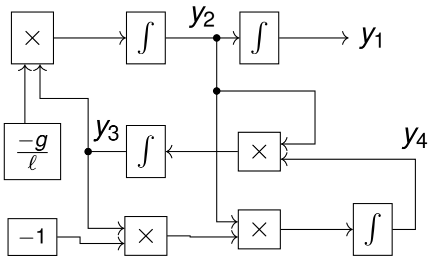

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Remark on “analog”

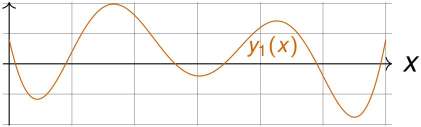
Continuous and analogy between circuits/mechanics/ODEs.

Computing with differential equations

Generable functions

$$\begin{cases} y(0) = y_0 \\ y'(x) = p(y(x)) \end{cases} \qquad x \in \mathbb{R}$$

$$f(x) = y_1(x)$$



Shannon's notion

sin, cos, exp, log, ...

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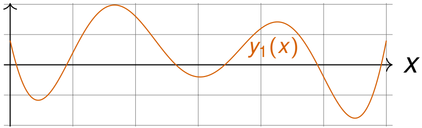

Shannon's notion

$\sin, \cos, \exp, \log, \ldots$

Considered "weak": not Γ and ζ
Only analytic functions

Computing with differential equations

Generable functions

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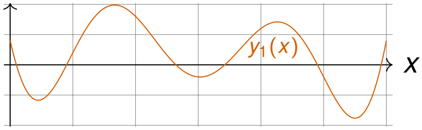

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Computable

$$\begin{cases} y(0) = q(x) & x \in \mathbb{R} \\ y'(t) = p(y(t)) & t \in \mathbb{R}_+ \end{cases}$$

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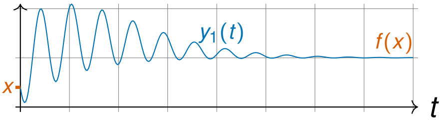

Modern notion

sin, cos, exp, log, Γ, ζ, ...

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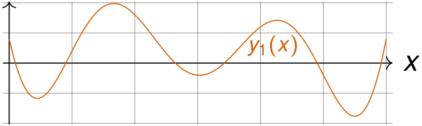

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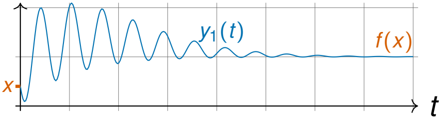
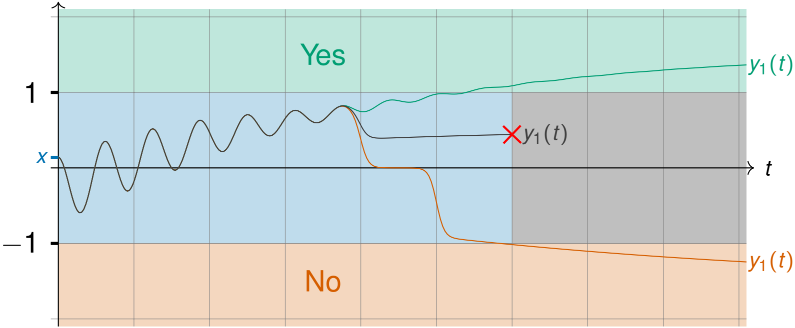


Modern notion

$\sin, \cos, \exp, \log, \Gamma, \zeta, \ldots$

Turing powerful
[Bournez et al., 2007]

More formally



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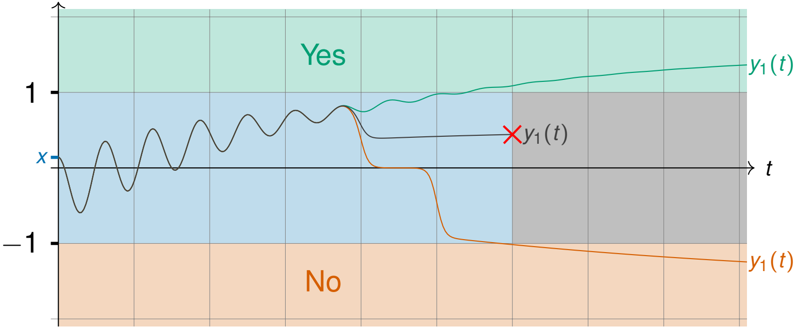
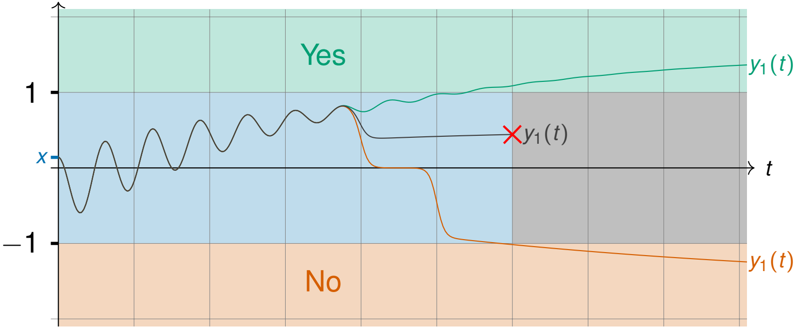


Theorem (Bournez et al, 2010)

This is equivalent to a Turing machine.

More formally



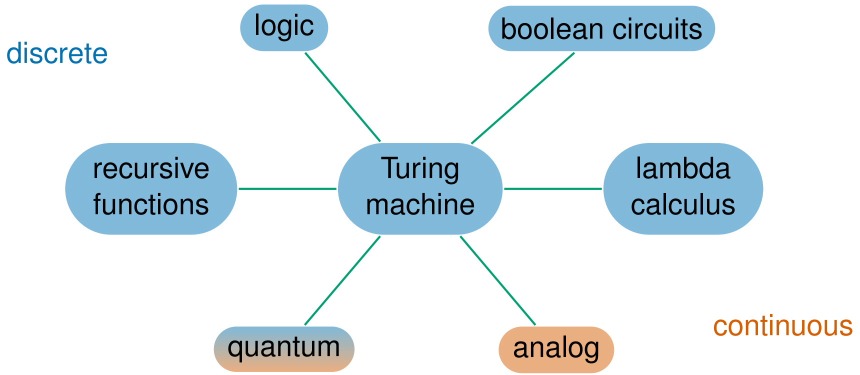
Theorem (Bournez et al, 2010)

This is equivalent to a Turing machine.

- analog computability theory
- purely continuous characterization of classical computability

Can Analog Machines Compute Faster?

Computability

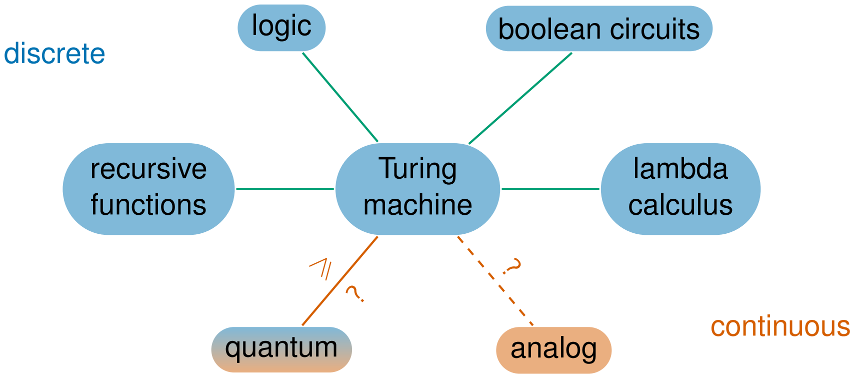


Church Thesis

All **reasonable** models of computation are equivalent.

Can Analog Machines Compute Faster?

Complexity



Effective Church Thesis

All **reasonable** models of computation are equivalent for complexity.

Complexity of analog systems

- Turing machines: $T(x) =$ number of steps to compute on x

Complexity of analog systems

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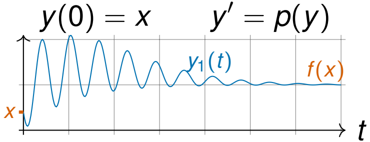
- GPAC:

Tentative definition

$T(x) = ??$



Complexity of analog systems

- Turing machines: $T(x) =$ number of steps to compute on x
- GPAC:

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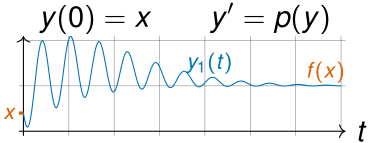
$T(x, \mu) =$



Complexity of analog systems

- Turing machines: $T(x) =$ number of steps to compute on x
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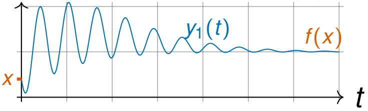
$T(x,\mu) =$ first time t so that $|y_1(t) - f(x)| \leqslant e^{-\mu}$

$y(0) = x \qquad y' = p(y)$



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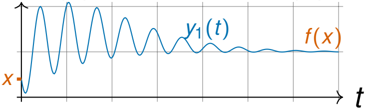
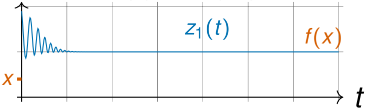


$\rightsquigarrow$

$z(t) = y(e^t)$



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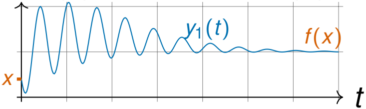

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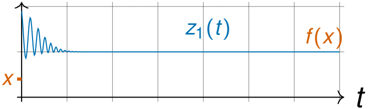
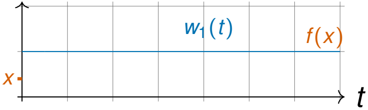


$w(t) = y(e^{e^t})$



Complexity of analog systems

- Turing machines: $T(x) =$ number of steps to compute on x
- GPAC: time contraction problem $\rightarrow$ **open problem**

Tentative definition

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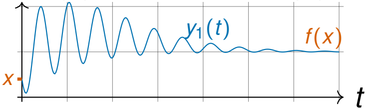

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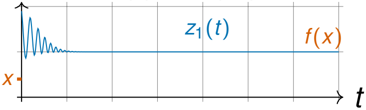

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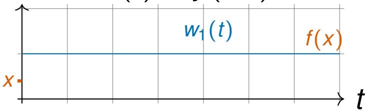
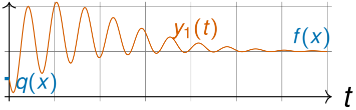


Something is wrong...

All functions have constant time complexity.

Time-space correlation of the GPAC

$y(0) = q(x) \qquad y' = p(y)$



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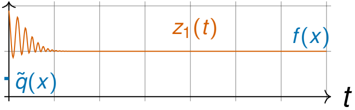

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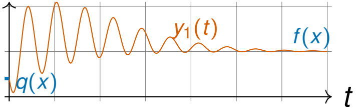

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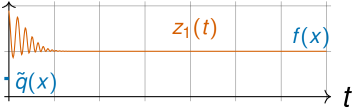
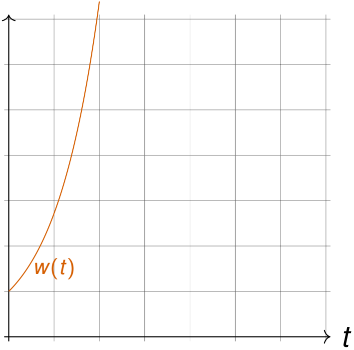


extra component: $w(t) = e^t$



Time-space correlation of the GPAC

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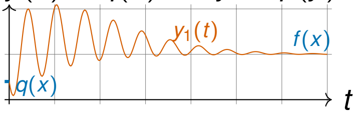

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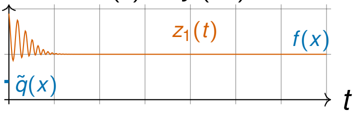

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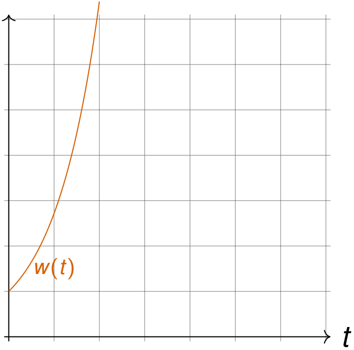


Observation

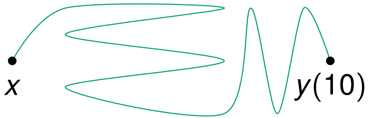
Time scaling costs "space".

$\rightsquigarrow$

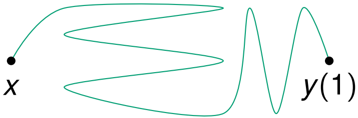
Time complexity for the GPAC must involve time and space !

Complexity in the analog world

Complexity measure: length of the curve



$=$



Time acceleration: same curve = same complexity !

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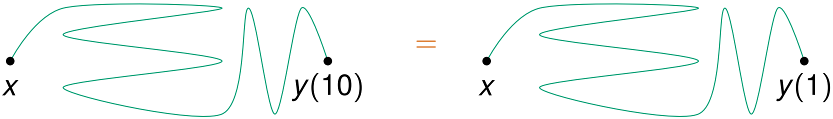
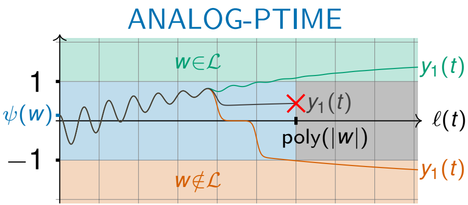
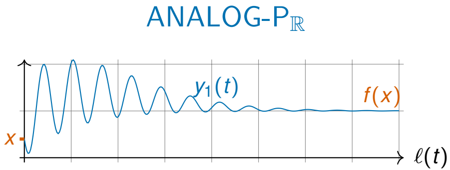


Time acceleration: same curve = same complexity !

Same time, different curves: different complexity !

Analog complexity





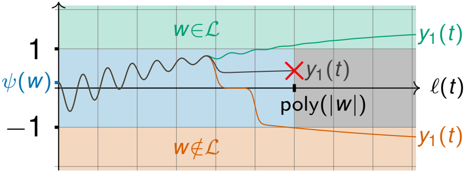
Theorem

- $\mathcal{L} \in$ PTIME *of and only if* $\mathcal{L} \in$ ANALOG-PTIME
- $f : [a, b] \to \mathbb{R}$ *computable in polynomial time* $\Leftrightarrow f \in$ ANALOG-$\mathrm{P}_{\mathbb{R}}$

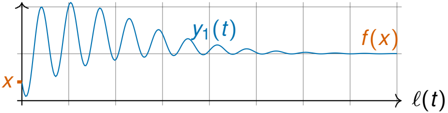
- Analog complexity theory based on length
- Time of Turing machine $\Leftrightarrow$ length of the GPAC
- Purely continuous characterization of PTIME

Analog complexity

ANALOG-PTIME



ANALOG-$P_{\mathbb{R}}$



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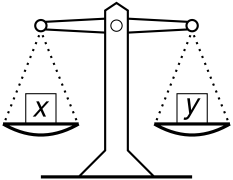
- Purely continuous characterization of PTIME
- Only rational coefficients needed

Does a balance scale compute a function?

Inputs: $x, y \in [0, +\infty)$



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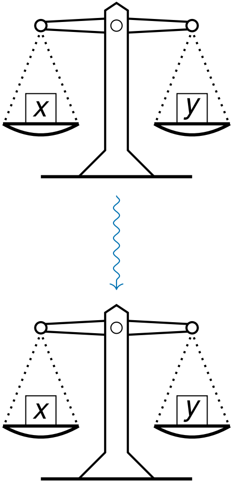

$x = y$

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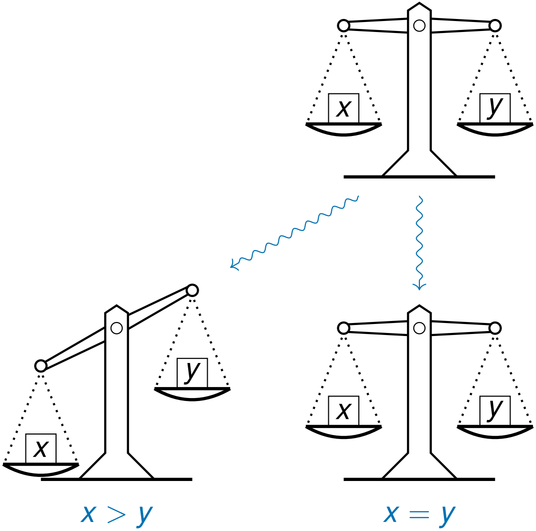

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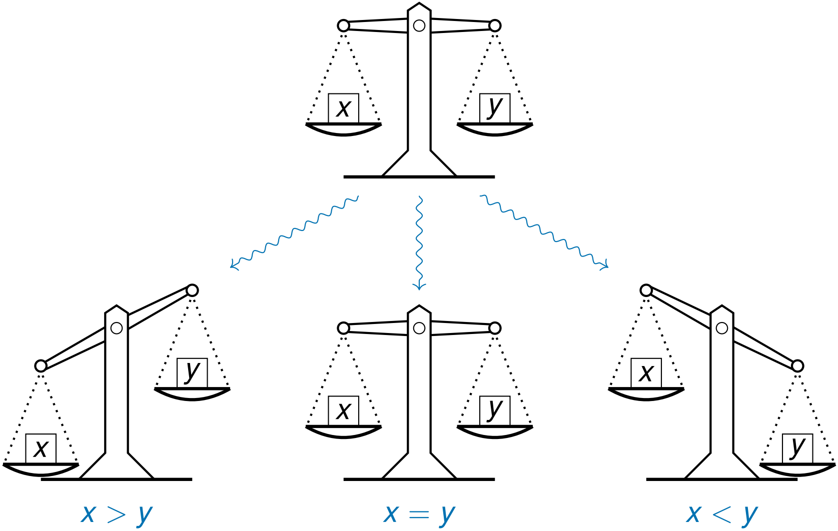

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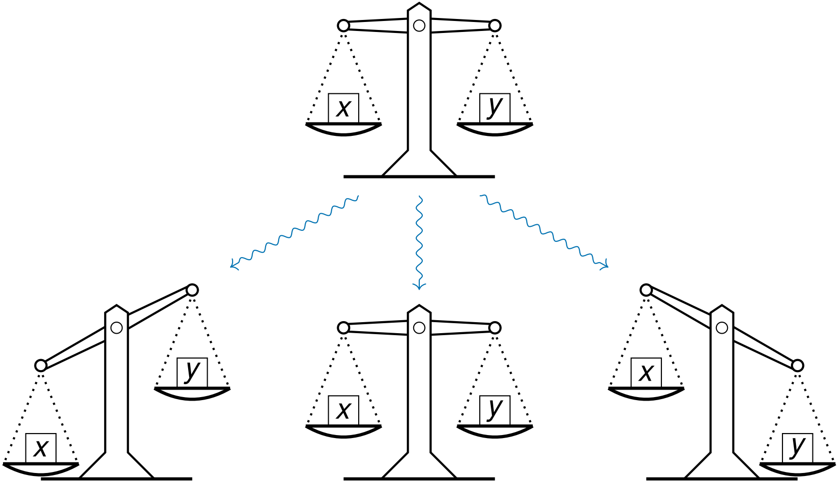

$x > y$ $x = y$ $x < y$

Output: $\mathrm{sign}(x - y)$?

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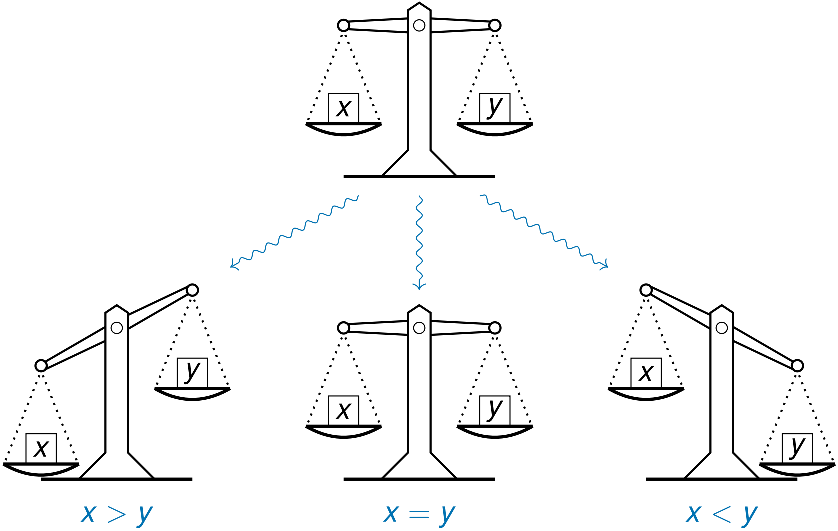
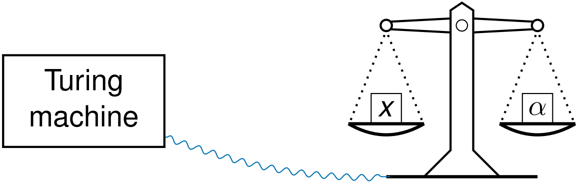


Output: $\mathrm{sign}(x - y)$? Complexity: ???

Physical Oracles (Beggs, Costa, Poças and Tucker)

Model: Turing Machine with “physical oracle”
Oracle: performs physical experiments with time limit
Outcomes: Yes, No, Timeout

Example: $\alpha \in [0, 1]$ unknown, x programmable



Queries:

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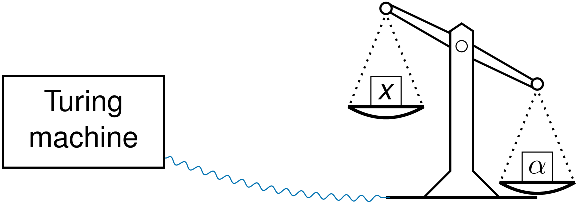

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- $x = \frac{1}{2}$, $T = 1$ second $\rightsquigarrow$ Yes

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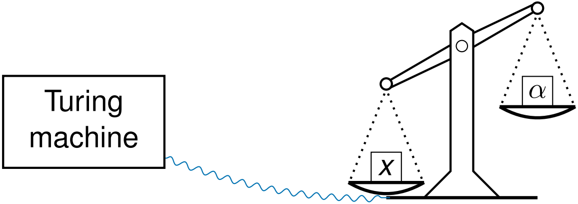

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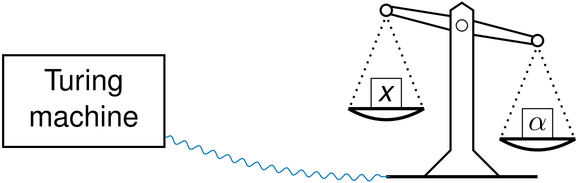

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- $x = \frac{5}{8}$, $T = 1$ second $\rightsquigarrow$ Timeout

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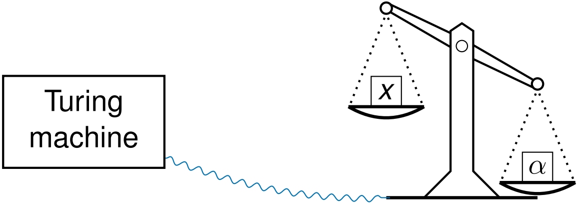

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- $x = \frac{5}{8}$, $T = 1$ second $\rightsquigarrow$ Timeout
- $x = \frac{5}{8}$, $T = 2$ seconds $\rightsquigarrow$ Yes

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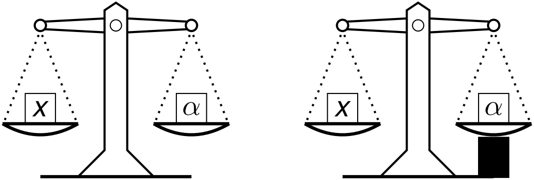

Wheatstone bridge
or
Brewster’s angle experiment

Experiments types:

- Two-sided: time $\frac{C}{|x-\alpha|^d}$, Yes/No/Timeout
- One-sided: time $\frac{C}{|x-\alpha|^d}$, Yes/Timeout
- Vanishing: Yes (if $x \neq \alpha$)/Timeout, can test if $|x-\alpha| \leqslant |x'-\alpha|$

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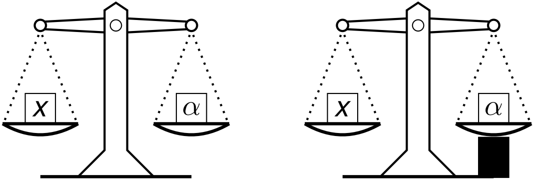

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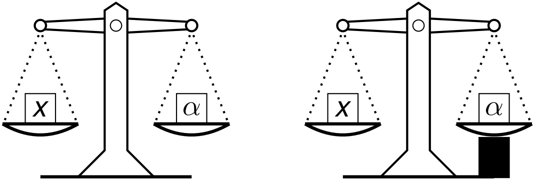
Precision: infinite, unbounded, fixed

Physical Oracles (Beggs, Costa, Poças and Tucker)



Wheatstone bridge
or
Brewster's angle experiment

Experiments types:

- Two-sided: time $\frac{C}{|x-\alpha|^d}$, Yes/No/Timeout
- One-sided: time $\frac{C}{|x-\alpha|^d}$, Yes/Timeout
- Vanishing: Yes (if $x \neq \alpha$)/Timeout, can test if $|x-\alpha| \leqslant |x'-\alpha|$

Precision: infinite, unbounded, fixed

Theorem (Beggs, Costa, Poças and Tucker)

For a broad class of oracles + PTIME machine, complexity bounded by[1] BPP//log$\star$, or P/poly if non-computable analog-digital interface.

[1] BPP + non-uniform log advise.

Summary

- analog: analogy/continuous, orthogonal meanings
- machines vs models: need to distinguish concepts
- “Church” thesis: subtle, several variants
- hypercomputability: various interpretations
- some reasonable models exists: GPAC, equivalent to TM
- complexity: difficult to define in general, several approaches
- influence of noise on computations