Discovering structure within the class of K-trivial sets

André Nies
mostly based on joint work with
Greenberg, Miller, Turetsky, in various combinations

August 26, 2020

THE UNIVERSITY OF AUCKLAND
NEW ZEALAND

Thanks to Noam for contributing source for 6 slides

Martin-Löf randomness

A central algorithmic randomness notion for infinite bit sequences is the one of Martin-Löf. There are several equivalent ways to define it. Here is one.

$Z \in 2^{\mathbb{N}}$ is Martin-Löf random $\iff$

for every computable sequence $(\sigma_i)_{i\in\mathbb{N}}$ of binary strings with $\sum_i 2^{-|\sigma_i|} < \infty$, there are only finitely many i such that σ_i is an initial segment of Z.

Note that $\lim_i 2^{-|\sigma_i|} = 0$, so this means that we cannot "Vitali cover" Z, viewed as a real number, with the collection of dyadic intervals corresponding to $(\sigma_i)_{i\in\mathbb{N}}$.

What does a ML-random compute?

- The Kučera-Gacs theorem says that each set $A \subseteq \mathbb{N}$ is Turing below some ML-random Z.
- If A is Δ^0_2, we can take Chaitin's Ω because $\Omega \equiv_T \emptyset'$

Conversely, if we are given a ML-random, which sets are Turing below it?

Theorem (Kučera 1985)

Each Δ^0_2 ML-random has a noncomputable c.e. set Turing below it.

Notation: ML stands for Martin-Löf.
MLR is the class of ML-random infinite bit sequences.

The randomness enhancement principle (N. 2010)

The less a ML-random Z computes, the more random it gets.

Example: Z is called weakly 2-random if Z is in no null Π^0_2 class. This is stronger than ML-randomness.

Weak 2-random $\Longleftrightarrow$ ML-random and forms a minimal pair with $\emptyset'$.

These results suggest a spectrum of randomness strength

- from ML-random (including examples such as Ω that computes all Δ^0_2 sets)
- to weakly 2-random (computing none but the computable sets).

Enter the K-trivials

Recall the Schnorr-Levin theorem:

- $Z \in 2^{\mathbb{N}}$ is ML-random if and only if $K(Z \upharpoonright n) \geq^{+} n$.

In the other extreme,

Definition (Chaitin, 1975)

$A \in 2^{\mathbb{N}}$ is K-trivial if $K(A \upharpoonright n) \leq^{+} K(n)$.

- computable $\Rightarrow$ K-trivial
- Chaitin: all K-trivials are Δ^0_2
- Solovay, '75: there is a noncomputable K-trivial set.

Letters A, B denote K-trivials. Letters Y, Z denote ML-randoms.

Characterisations of K-trivials

Theorem (Nies-Hirschfeldt;Nies 2003)

The following are equivalent for $A \in 2^{\mathbb{N}}$:

1. A is K-trivial.
2. $K^A =^+ K$ (A is low for K).
3. $\mathsf{MLR}^A = \mathsf{MLR}$ (A is low for ML-randomness).

Theorem (Nies 2003)

1. K-triviality is Turing-invariant.
2. The K-trivial Turing degrees form an ideal contained in the superlow sets.
3. Every K-trivial set is Turing below a c.e. K-trivial set.

Basis for randomness

Theorem (Hirschfeldt, Nies, Stephan, 2006)

$A \in 2^{\mathbb{N}}$ is K-trivial if and only if $A \leq_T Z$ for some $Z \in \mathsf{MLR}^A$.

Left to right follows from the equivalence of K-triviality with lowness for ML-randomness, and the Kučera-Gacs Theorem.

Proposition (Hirschfeldt, Nies, Stephan, 2006)

If $A \leq_T Z$ where A is c.e. and Z is ML-random with $\emptyset' \not\leq_T Z$, then $Z \in \mathsf{MLR}^A$. And hence A is K-trivial.

- In other words, if A is c.e. and NOT K-trivial, then any ML-random $Z \geq_T A$ is above $\emptyset'$.
- So there is no version of Kučera-Gacs within the Turing incomplete sets.

Characterising the c.e. K-trivials in terms of plain ML-randomness and computability notions

We've seen that every c.e. set below an incomplete ML-random is K-trivial. The converse stayed open for a while.

Theorem (Bienvenu, Greenberg, Kucera, N., Turetsky '16 & Day, Miller, '16)

The following are equivalent for a c.e. set:

- A is computable from some incomplete ML-random;
- A is K-trivial.

And in fact, there is a single incomplete Δ^0_2 ML-random above all the K-trivials!

ML-reducibility

By 2016 there were 17 or so characterisations of the class, but little was known about the internal structure of the K-trivials:

They form an ideal in the Turing degrees that is contained in superlow, generated by its c.e. members, and has no greatest degree (nonprincipal).

It turns out that Turing reducibility $\leq_T$ is too fine to understand the structure. We are standing too close to see the structure. A coarser "reducibility" is suggested by the results above.

Definition (main for this talk)

For sets A, B, we write $B \geq_{ML} A$ if

$$\forall Z \in \mathsf{MLR}\ [Z \geq_T B \Rightarrow Z \geq_T A].$$

(Any ML-random computing B also computes A.)

Recall: For sets A, B, we write $B \geq_{ML} A$ if $\forall Z \in \mathsf{MLR}[Z \geq_T B \Rightarrow Z \geq_T A]$.

- A common paradigm: computational lowness means to be not overly useful as an oracle. $\leq_{LR}$ and other weak reducibilities are based on this. Later on we will introduce $\leq_{SJT}$, a weakening of $\leq_T$, also following this paradigm.
- ML-reducibility seeks to understand relative complexity of sets via an alternative lowness paradigm: computational lowness means being computed by many oracles.

Some facts

- By HNS 06, the ML-degree of $\emptyset'$ contains all the non-K-trivial c.e. sets. So among the c.e. sets one can focus on K-trivials.
- Each K-trivial A is ML-equivalent to a c.e. K-trivial $D \geq_T A$. (GMNT, arXiv 1707.00258)

Structure of the K-trivials w.r.t. $\leq_{ML}$

- The least degree consists of the computable sets (by the low basis theorem with upper cone avoiding).
- There is a ML-complete K-trivial, called a "smart" K-trivial. (BGKNT, JEMS 2016)
- There is a dense hierarchy of principal ideals $\mathcal{B}_q$, $q \in (0, 1)_{\mathbb{Q}}$. E.g., $\mathcal{B}_{0.5}$ consists of the sets that are computed by both "halves" of a ML-random Z, namely Z_{even} and Z_{odd} (GMN, JML 2019)
- further interesting subclasses of the K-trivials are downward closed under $\leq_{ML}$.
- E.g. the strongly jump traceables, or equivalently the sets below all the ω-c.a. ML-randoms (by HGN, Adv. Maths 2012, along with GMNT).

A bit of degree theory for $\leq_{ML}$ on the K-trivials

Recall: $B \geq_{ML} A$ if $\forall Z \in \mathsf{MLR}[Z \geq_T B \Rightarrow Z \geq_T A]$.

Results from GMNT, arxiv 1707.00258

- For each noncomputable c.e. K-trivial D there are c.e. $A, B \leq_T D$ such that $A \mid_{ML} B$.
- No minimal pairs.
- For each c.e. A there is c.e. $B >_T A$ such that $B \equiv_{ML} A$.

The first is based on a method of Kučera. The second and third use cost functions.

Density? No idea.

The problem is we don't even know whether $\leq_{ML}$ is arithmetical. So it's hard to envisage a construction showing density. The results also hold for the even coarser, but arithmetical reducibility where Z is restricted to the Δ^0_2 sets. Density may be easier to show there.

Cost functions

Definition

A cost function is a computable function $\mathbf{c}\colon \mathbb{N}^2 \to \mathbb{R}^{\geq 0}$.

Extra requirements: monotonicity ($\mathbf{c}(x,s) \geq \mathbf{c}(x+1,s)$ and $\mathbf{c}(x,s) \leq \mathbf{c}(x,s+1)$); finiteness (for all x, $\underline{\mathbf{c}}(x) = \lim_s \mathbf{c}(x,s)$ exists); the limit condition ($\lim_x \underline{\mathbf{c}}(x) = 0$).

Definition

Let $\langle A_s \rangle$ be a computable approximation of a Δ^0_2 set A; let $\mathbf{c}$ be a cost function. The total cost $\mathbf{c}(\langle A_s \rangle)$ is

$$\sum_s \mathbf{c}(x,s)[\![x \text{ is least s.t. } A_s(x) \neq A_{s-1}(x)]\!].$$

A Δ^0_2 set A obeys a cost function $\mathbf{c}$ if there is some computable approximation $\langle A_s \rangle$ of A for which the total cost $\mathbf{c}(\langle A_s \rangle)$ is finite.

Write $A \models \mathbf{c}$ for this. There is a c.e. $A \models \mathbf{c}$.

Cost functions characterising ML-ideals

Recall: a Δ^0_2 set obeys $\mathbf{c}$ if it can be computably approximated obeying the "speed limit" given by $\mathbf{c}$.

Let $\mathbf{c}_\Omega(x, s) = \Omega_s - \Omega_x$ (where $\langle \Omega_s \rangle$ is an increasing approximation of Ω).

Theorem (N., Calculus of cost functions, 2017)

A Δ^0_2 set is K-trivial if and only if it obeys $\mathbf{c}_\Omega$.

Let $\mathbf{c}_{\Omega,1/2}(x, s) = (\Omega_s - \Omega_x)^{1/2}$.

Theorem (GMN, 2019)

The following are equivalent:

1. A is computed by both halves of a ML-random.
2. A obeys $\mathbf{c}_{\Omega,1/2}$.

Cost functions and computing from randoms

Definition

Let $\mathbf{c}$ be a cost function. A $\mathbf{c}$-test is a sequence (U_n) of uniformly Σ^0_1 subsets of $\{0,1\}^{\mathbb{N}}$ satisfying $\lambda(U_n) = O(\underline{\mathbf{c}}(n))$.

Main Fact

Suppose that $Z \in \mathsf{MLR}$ is captured by a $\mathbf{c}$-test, and A obeys $\mathbf{c}$. Then $A \leq_T Z$.

Proof idea: Collect the oracles that may become invalid through A-change into a Solovay test.

If an approximation of A obeying $\mathbf{c}$ changes $A(n)$ at stage s, then $U_{n,s}$ is listed as a component of the test.

Z is outside almost all components, so Z computes A correctly a.e.

Definition (ML-completeness for a cost function, GMNT)

Let $\mathbf{c} \geq \mathbf{c}_\Omega$ be a cost function. We say that a K-trivial A is smart for $\mathbf{c}$ if A is ML-complete among the sets that obey $\mathbf{c}$.

Thus $A \models \mathbf{c}$ and $B \leq_{ML} A$ for each $B \models \mathbf{c}$.

Theorem (GMNT, extending BGKNT result for $\mathbf{c}_\Omega$)

For each $\mathbf{c} \geq \mathbf{c}_\Omega$ there is a c.e. set A that is smart for $\mathbf{c}$.

May assume $\mathbf{c}(k) \geq 2^{-k}$. Build A. There is a particular Turing functional Γ such that it suffices to show $A = \Gamma^Y \Rightarrow Y$ fails some $\mathbf{c}$-test.

- During construction, let $\mathcal{G}_{k,s} = \{Y : \Gamma_t^Y \upharpoonright 2^{k+1} \prec A_t \text{ for some } k \leq t \leq s\}$.
- Error set $\mathcal{E}_s$: those Y such that Γ_s^Y is to the left of A_s.
- Ensure $\lambda \mathcal{G}_{k,s} \leq \mathbf{c}(k,s) + \lambda(\mathcal{E}_s - \mathcal{E}_k)$. If this threatens to fail put next $x \in [2^k, 2^{k+1})$ into A. Then $\langle \mathcal{G}_k \rangle$ is the required $\mathbf{c}$-test.

ML-completeness for a cost function

Definition (recall)

Let $\mathbf{c} \geq \mathbf{c}_\Omega$ be a cost function. We say that a K-trivial A is smart for $\mathbf{c}$ if A is ML-complete among the sets that obey $\mathbf{c}$.

Theorem (GMNT)

For each K-trivial A there is a cost function $\mathbf{c}_A \geq \mathbf{c}_\Omega$ such that A is smart for $\mathbf{c}_A$.

This shows that there are no ML-minimal pairs:
if K-trivials A, B are noncomputable, there is a noncomputable c.e. D such that $D \models \mathbf{c}_A + \mathbf{c}_B$.
Then $D \leq_{ML} A, B$.

Smartness and half-bases

Recall:

Theorem (BGKNT)

Not every K-trivial is a half-base.

Proof.

- Ω_{even} and Ω_{odd} are low;
- If $Y \in \mathsf{MLR}$ is captured by a $\mathbf{c}_\Omega$-test, then it is superhigh.
- So a smart K-trivial is not a half-base.

$\square$

Dual reducibility

Definition

For $Z, Y \in \mathsf{MLR}$, let $Z \leq_{ML^*} Y$ if for every K-trivial A,

$$A \leq_T Z \quad \Rightarrow \quad A \leq_T Y.$$

We say that $Z \in \mathsf{MLR}$ is feeble for $\mathbf{c}$ if Z is captured by a $\mathbf{c}$-test and has least ML^*-degree among those. For example:

- For rational $p \in (0, 1)$, any appropriate "p-part" of Ω is feeble for $\mathbf{c}_{\Omega,p}$.
- Top degree: all randoms captured by a $\mathbf{c}_\Omega$-test (non Oberwolfach-randoms).
- Bottom degree: the weakly 2-randoms.

Pieces of Ω w.r.t. $\leq_{ML^*}$

- For any infinite computable $R \subseteq \mathbb{N}$, let Ω_R be the bits of Ω with position in R.
- We can define a corresponding cost function $\mathbf{c}_{\Omega,R}$ similar to $\mathbf{c}_{\Omega,p}$: A obeys $\mathbf{c}_{\Omega,R} \iff A \leq_T \Omega_R$.
- Thus, Ω_R is feeble for $\mathbf{c}_{\Omega,R}$.

For each R, let B_R be a K-trivial smart for $\mathbf{c}_{\Omega_R}$.

Theorem (GMNT)

The following are equivalent for infinite, computable $R, S \subseteq \mathbb{N}$:

1. $\Omega_S \leq_{ML^*} \Omega_R$;
2. $B_S \geq_{ML} B_R$;
3. $|S \cap n| \leq^+ |R \cap n|$.

For instance, by (3), Ω_{even} and Ω_{odd} compute the same K-trivials!

Other weak reducibilities

- Note that $A \leq_T B$ if $J^A = \Psi^B$ for some functional Ψ (where $J^X = \phi_e^X(e)$ is the jump of X).
- Suppose B instead only can make a small, small number of guesses for $J^A(x)$?

Definition (N. 2009; related to Cole and Simpson 06)

We write $A \leq_{SJT} B$ if for each order function h, there is a uniform list $\langle \Psi_r \rangle$ of functionals such that $J^A(x)$, if defined, equals $\Psi_r^B(x)$ for some $r \leq h(x)$.

- A is strongly jump traceable (FNS 05) if $A \leq_{SJT} \emptyset$. These sets are properly contained in the K-trivials.
- There is no $\leq_{SJT}$-largest K-trivial, essentially by relativizing this.

Recall that Y is ω-c.a. if $Y \leq_{\mathrm{wtt}} \emptyset'$.
Let $\mathcal{C}$ be the class of the ω-c.a., superlow, or superhigh sets.

Theorem (Work in progress with Greenberg and Turetsky)

The following are equivalent for K-trivial c.e. sets A, B.

(a) $A \leq_{SJT} B$

(b) $A \leq_T B \oplus Y$ for each $Y \in \mathcal{C} \cap \mathsf{MLR}$.

This generalises work of [GHN 2012] where $B = \emptyset$.
So we have on the K-trivials that

$$\leq_T \quad \Rightarrow \quad \leq_{ML} \quad \Rightarrow \quad \leq_{\omega-\text{c.a.}-ML}$$

$$\leq_T \quad \Rightarrow \quad \leq_{SJT} \quad \Rightarrow \quad \leq_{\omega-\text{c.a.}-ML}$$

and none of $\leq_{ML}$, $\leq_{SJT}$ implies the other.

Questions

- Is being a smart K-trival an arithmetical property? Stronger: is $\leq_{\mathrm{ML}}$ an arithmetical relation?
- Are the ML-degrees of the K-trivials dense?
- Can a smart K-trivial be cappable? Can it obey a cost function much stronger than $\mathbf{c}_\Omega$?
- Is there an incomplete ω-c.a. ML-random above all the K-trivials?

Some references

- Bienvenu, Greenberg, Kučera, Nies, Turetsky: Coherent randomness tests and computing the K-trivial sets, JEMS 2016
- Greenberg, J. Miller, Nies: Computing from projections of random points, JML 2019
- Greenberg, J. Miller, Nies, Turetsky: Martin-Löf reducibility and cost functions. Early version arxiv 1707.00258