The characterization of Weihrauch reducibility in systems containing $\text{E-PA}^\omega + \text{QF-AC}^{0,0}$

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Motivation

We represent problems as formulas:

$$P := \forall x(\underbrace{A(x)}_{\text{domain}} \to \exists y\, \underbrace{B(x,y)}_{\text{matrix}}).$$

Can we find a system of (at least) second-order arithmetic $\mathcal{A}$ and a calculus $\mathcal{C}$ such that the following holds for two problems P and Q?

$$\mathcal{A} \vdash \text{"}Q \leq_W P\text{"}$$

$$\Leftrightarrow$$

$$\mathcal{C} \vdash P' \to Q'.$$

Results in this direction

Theorem (Hirst and Mummert 2019)

Suppose P and Q are nice problems of the form

$$P := \forall x(A(x) \to \exists y B(x,y))$$
$$Q := \forall u(C(u) \to \exists v D(u,v)).$$

then the following are equivalent:

a) $i\,\mathsf{RCA}_0^\omega$ *proves Q with one typical use of P,*

b) $i\,\mathsf{RCA}_0^\omega \vdash Q \leq_W P$.

Theorem (Fujiwara 2020)

Several characterization results of Weihrauch reducibility in $\mathsf{E\text{-}PA}^\omega$ / $\widehat{\mathsf{E\text{-}PA}}^\omega \restriction + \mathsf{AC}^\omega$ / $\Pi^0_1\text{-}\mathsf{AC}^{0,0}$ / $\mathsf{QF\text{-}AC}^{0,0}$.

Both results rely on a special proof structure

A first approach

Theorem 6.4 (Kuyper JSL 2017)

Characterizes compositional Weihrauch reducibility in RCA_0 using EL_0 (elementary intuitionistic analysis)$+$ MP (Markov's principle).

Theorem 7.1 (Kuyper JSL 2017)

Characterizes Weihrauch reducibility in RCA_0 using $(\mathrm{EL}_0 + \mathrm{MP})^{\exists\alpha a}$ that is defined like $\mathrm{EL}_0 + \mathrm{MP}$ but

- contraction is only allowed for formulas without function quantifiers and
- weakening is only allowed for subformulas of $\exists\alpha A$ where A does not contain function quantifiers.

Counterexamples (Uftring M.Sc. thesis 2018)

But the general idea seems to be correct.

The goal

Consider

$$P :\equiv \forall x^1 (A(x) \to \exists y^1 B(x,y))$$
$$Q :\equiv \forall u^1 (C(u) \to \exists v^1 D(u,v))$$

Theorem (Simplified)

The following are equivalent:

a) $\text{E-LPA}^\omega_\ell + \Gamma^\bullet$ *proves* $P' \multimap Q'$

b) $\text{E-PA}^\omega + \text{QF-AC}^{0,0} + \Gamma$ *proves* $Q \leq_W P$

Linear Logic

Every formula is a resource

Symbols of linear logic

- Conjunctions: $A \otimes B$, $A \mathbin{\&} B$
- Disjunctions: $A \parr B$, $A \oplus B$
- Modal: $!A$, $?A$
- Involution: $A^\perp$
- Abbreviation: $(A \multimap B) :\equiv A^\perp \parr B$

Embedding of classical logic into linear logic

$$
\begin{aligned}
A^\bullet &:\equiv A \text{ where } A \text{ is atomic,} \\
(\neg A)^\bullet &:\equiv (A^\bullet)^\perp, \\
(A \wedge B)^\bullet &:\equiv A^\bullet \otimes B^\bullet, \\
(A \vee B)^\bullet &:\equiv A^\bullet \parr B^\bullet, \\
(A \to B)^\bullet &:\equiv A^\bullet \multimap B^\bullet.
\end{aligned}
$$

Linear Logic (Intuition)

Every argument must be used exactly once:

Examples

$\vdash A \otimes B \multimap B \otimes A$	
$\vdash A \multimap (B \multimap A \otimes B)$	
$\nvdash A \multimap A \otimes A$	We cannot simply multiply A.
$\vdash !A \multimap A \otimes A$	We may use $!A$ as often as we like.
$\nvdash A \otimes B \multimap A$	We must use B.
$\vdash A \otimes !B \multimap A$	We may choose to use $!B$ not at all.

Dualities

$$(A \otimes B)^{\perp} \equiv A^{\perp} \parr B^{\perp}$$

$$(!A)^{\perp} \equiv ?A^{\perp}$$

Connectives $\parr$ and "?" do not have a simple intuition.

Motivating a linear predicate

Problem: Quantifiers in problems cause problems
Solution: Proof theory on nonstandard arithmetic
(van den Berg, Briseid, Safarik 2012)

Standard Predicate

- $\mathrm{st}(x) \wedge x = y \to \mathrm{st}(y)$
- $\mathrm{st}(t_c)$ where t_c is closed
- $\mathrm{st}(f) \wedge \mathrm{st}(x) \to \mathrm{st}(fx)$
- $\Phi(0) \wedge \forall^{\mathrm{st}} n^0 (\Phi(n) \to \Phi(n+1)) \to \forall^{\mathrm{st}} n^0 \Phi(n)$

Nonstandard Dialectica only extracts information about standard values.

Idea: Adapt this predicate to linear logic

- Only extract information about the Weihrauch reduction
- Uniform extraction that works with problems involving quantifiers

E-LPA$^\omega_\ell$

Extensional Linear Peano Arithmetic in all finite types with linear predicate consists of the following three parts:

- The axioms and rules of linear logic,
- The axioms of E-PA$^\omega$ translated to linear logic,
- Additional axioms for the new linear predicate ℓ:

$$\vdash \ell(t_c) \qquad \vdash A_{nl}^\perp, !A_{nl} \qquad \vdash \ell(t) \multimap \ell(t) \otimes \ell(t)$$

$$\vdash \ell^\perp(t), \ell^\perp(r), \ell(tr)$$

$$\vdash (\forall x^0 \exists y^0 \alpha xy =_0 0)^\perp, \exists Y^1 (\forall x^0 (\alpha x(Yx) =_0 0) \otimes !(\ell(\alpha) \multimap \ell(Y)))$$

Abbreviations:

$$\forall^\ell x A :\equiv \forall x(\ell(x) \multimap A)$$
$$\exists^\ell x A :\equiv \exists x(\ell(x) \otimes A)$$
$$\exists^\ell_\epsilon x A :\equiv \exists x(\ell(x) \otimes \epsilon =_0 0 \otimes A)$$

For $\epsilon := 0$ and $\epsilon := 1$, $\exists^\ell_\epsilon x A$ behaves like $\exists^\ell x A$ and $\perp$, respectively.

Formalization of Weihrauch reducibility

Problems

$$P :\equiv \forall x^1(A(x) \to \exists y^1 B(x,y))$$
$$Q :\equiv \forall u^1(C(u) \to \exists v^1 D(u,v))$$

In E-LPA$^\omega_\ell$

$$P' :\equiv \forall^\ell x^1(A^\bullet(x) \multimap \exists^\ell_\epsilon y^1 B^\bullet(x,y))$$
$$Q' :\equiv \forall^\ell u^1(C^\bullet(u) \multimap \exists^\ell_\epsilon v^1 D^\bullet(u,v))$$

Weihrauch reducibility formalized using associates

There are closed terms t and s such that the formulas

$$\forall u^1(C(u) \to t \cdot u{\downarrow} \wedge A(t \cdot u))$$

and $\forall u^1, y^1(C(u) \wedge B(t \cdot u, y) \to s \cdot j(u,y){\downarrow} \wedge D(u, s \cdot j(u,y)))$

hold.

The Characterization of Weihrauch reducibility

Theorem (Uftring 2018, 2020)

Let $A(x^1)$, $B(x, y^1)$, $C(u^1)$, and $D(u, v^1)$ be formulas of $\mathrm{E\text{-}PA}^\omega$. Let Γ be a set of formulas of the same language. Consider:

$$\vdash \forall^\ell x^1 (A^\bullet(x) \multimap \exists^\ell_\epsilon y^1 B^\bullet(x, y)) \multimap \forall^\ell u^1 (C^\bullet(u) \multimap \exists^\ell_\epsilon v^1 D^\bullet(u, v)).$$

The following are equivalent:

a) $\mathrm{E\text{-}LPA}^\omega_\ell + \Gamma^\bullet$ *proves the sequent.*

b) $\mathrm{E\text{-}APA}^\omega_\ell + \Gamma^\bullet$ *proves the sequent.*

c) $\mathrm{E\text{-}PA}^\omega + \mathrm{QF\text{-}AC}^{0,0} + \Gamma$ *proves both*

$$C(u) \to t \cdot u{\downarrow} \wedge A(t \cdot u)$$

and

$$C(u) \wedge B(t \cdot u, y) \to s \cdot j(u, y){\downarrow} \wedge D(u, s \cdot j(u, y))$$

for some closed terms t^1 and s^1 of $\mathcal{L}(\mathrm{E\text{-}PA}^\omega)$.

Gödel's Dialectica interpretation for linear logic

Inspired by work due to de Paiva (1991), Shirahata (2006), and Oliva (2008-2011):

$$
\begin{aligned}
|A| &:\equiv A \text{ for unnegated + nonlinear atomic } A,\\
|A^{\perp}|^{\mathsf{u}}_{\mathsf{v}} &:\equiv (|A|^{\mathsf{v}}_{\mathsf{u}})^{\perp} \text{ for unnegated atomic } A,\\
|A \oplus B|^{\mathsf{x},\mathsf{u},k^0}_{\mathsf{y},\mathsf{v}} &:\equiv (!k =_0 0 \otimes |A|^{\mathsf{x}}_{\mathsf{y}}) \oplus (!k \neq_0 0 \otimes |B|^{\mathsf{u}}_{\mathsf{v}}),\\
|A \,\&\, B|^{\mathsf{x},\mathsf{u}}_{\mathsf{y},\mathsf{v},k^0} &:\equiv (!k =_0 0 \multimap |A|^{\mathsf{x}}_{\mathsf{y}}) \,\&\, (!k \neq_0 0 \multimap |B|^{\mathsf{u}}_{\mathsf{v}}),\\
|A \parr B|^{\mathsf{f},\mathsf{g}}_{\mathsf{x},\mathsf{u}} &:\equiv |A|^{\mathsf{fu}}_{\mathsf{x}} \parr |B|^{\mathsf{gx}}_{\mathsf{u}},\\
|A \otimes B|^{\mathsf{x},\mathsf{u}}_{\mathsf{f},\mathsf{g}} &:\equiv |A|^{\mathsf{x}}_{\mathsf{fu}} \otimes |B|^{\mathsf{u}}_{\mathsf{gx}},\\
|\exists z A|^{\mathsf{x}}_{\mathsf{y}} &:\equiv \exists z |A|^{\mathsf{x}}_{\mathsf{y}},\\
|\forall z A|^{\mathsf{x}}_{\mathsf{y}} &:\equiv \forall z |A|^{\mathsf{x}}_{\mathsf{y}},\\
|?A|_{\mathsf{y}} &:\equiv ?\exists \mathsf{x} |A|^{\mathsf{x}}_{\mathsf{y}},\\
|!A|^{\mathsf{x}} &:\equiv !\forall \mathsf{y} |A|^{\mathsf{x}}_{\mathsf{y}}.
\end{aligned}
$$

Biggest modification: Quantified values are not interpreted

Interpretation of the linear predicate

Interpreting the standard predicate (simplified)

$$|\mathrm{st}(t)|^x :\equiv x = t$$

Constructing a term

$$\vec{0} :\equiv 1,$$
$$\overrightarrow{(\tau\rho)} :\equiv \vec{\tau}\vec{\rho}.$$

Hereditary version of associates (Kleene, Kreisel 1959)

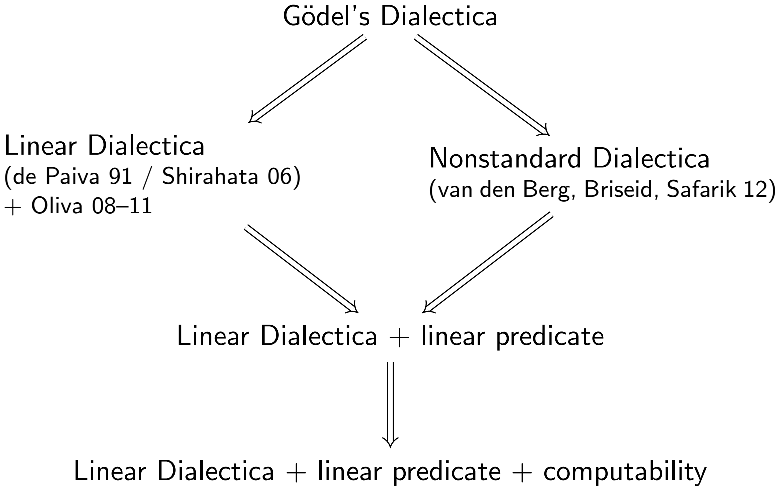
$$\begin{aligned}
\mathrm{con}_0(s^1, t^0) &:\equiv \exists x^0(sx \neq_0 0) \wedge \forall x^0(sx \neq_0 0 \to sx =_0 t+1),\\
\mathrm{con}_{\tau\rho}(s^{\vec{\tau}\vec{\rho}}, t^{\tau\rho}) &:\equiv \forall x^{\vec{\rho}}, y^{\rho}(\mathrm{con}_{\rho}(x,y) \to \mathrm{con}_{\tau}(sx, ty)).
\end{aligned}$$

Theorem: For each closed term t there is some $\tilde{t}$ with $\mathrm{con}(\tilde{t}, t)$.

Interpreting the linear predicate

$$|\ell(t)|^x :\equiv \mathrm{con}^\bullet(x, t)$$

“History” of our functional interpretation



Soundness Theorem of Dialectica for E-LPA$^\omega_\ell$

Theorem

Let A_1, ..., A_n be formulas of $\mathcal{L}$(E-LPA$^\omega_\ell$), and Γ a set of formulas in $\mathcal{L}$(E-PA$^\omega$), and assume that E-LPA$^\omega_\ell$ $+\Gamma^\bullet$ (or E-APA$^\omega_\ell$ $+\Gamma^\bullet$) proves

$$\vdash A_1, \ldots, A_n.$$

then E-LPA$^\omega_\ell$ $+\Gamma^\bullet$ (or E-APA$^\omega_\ell$ $+\Gamma^\bullet$) proves

$$\vdash |A_1|^{\mathsf{a}_0}_{\mathsf{x}_0}, \ldots, |A_n|^{\mathsf{a}_n}_{\mathsf{x}_n}$$

for tuples of terms a_0, ..., a_n where the free variables of each a_i are among those in the sequence of terms $\mathsf{x}_0, \ldots, \mathsf{x}_{i-1}, \mathsf{x}_{i+1}, \ldots, \mathsf{x}_n$. In particular, the variables x_i are not free in a_i.

Proof.

Induction on the proof length, i.e., for all rules. □

Proof sketch for the Characterization Theorem

Given a proof of the following in E-LPA$^\omega_\ell$ $+\Gamma^\bullet$ $+$ QF-AC^{0,0}:

$$\vdash \forall^\ell x^1(A^\bullet(x) \multimap \exists^\ell_\epsilon y^1 B^\bullet(x,y)) \multimap \forall^\ell u^1(C^\bullet(u) \multimap \exists^\ell_\epsilon v^1 D^\bullet(u,v)).$$

"$\epsilon := 1$"

$$\vdash \forall^\ell x^1(A^\bullet(x) \multimap \bot)) \multimap \forall^\ell u^1(C^\bullet(u) \multimap \bot).$$

Extract term t' mapping each $\tilde{u}$ with $C(u)$ to an $\tilde{x}$ with $A(x)$
$\Rightarrow$ Associate t computing for each u with $C(u)$ an x with $A(x)$

"$\epsilon := 0$" $+$ previous result

$$\vdash \forall^\ell u^1(\exists^\ell y^1 B^\bullet(t \cdot u, y) \multimap C^\bullet(u) \multimap \exists^\ell v^1 D^\bullet(u,v)).$$

Extract term s' mapping each $\tilde{u}$, $\tilde{y}$ with $B(t \cdot u, y)$ and $C(u)$ to $\tilde{v}$ with $D(u,v)$.
$\Rightarrow$ Associate s computing for each u and y with $B(t \cdot u, y)$ and $C(u)$ a v with $D(u,v)$.

Associates t and s compute the Weihrauch reduction in E-PA$^\omega$ $+\Gamma$ □

The Characterization of Weihrauch reducibility (pretty)

Theorem (Uftring 2020)

Let $A(x^1)$, $B(x, y^1)$, $C(u^1)$, and $D(u, v^1)$ be formulas of E-PA^ω*.* *Let Γ be a set of formulas of the same language. Consider:*

$$\vdash \forall^\ell x^1(A^\bullet(x) \multimap \exists^\ell y^1 B^\bullet(x, y)) \multimap \forall^\ell u^1(C^\bullet(u) \multimap \exists^\ell v^1 D^\bullet(u, v)).$$

The following are equivalent:

a) $\text{E-LPA}^\omega_\ell + \Gamma^\bullet$ *proves the sequent.*

b) ~~$\text{E-APA}^\omega_\ell + \Gamma^\bullet$ *proves the sequent.*~~

c) $\text{E-PA}^\omega + \text{QF-AC}^{0,0} + \Gamma$ *proves both*

$$C(u) \to t \cdot u{\downarrow} \wedge A(t \cdot u)$$

and $$C(u) \wedge B(t \cdot u, y) \to s \cdot j(u, y){\downarrow} \wedge D(u, s \cdot j(u, y))$$

for some closed terms t^1 and s^1 of $\mathcal{L}(\text{E-PA}^\omega)$*.*

Making the result more pretty

What happens if we use $\exists^\ell$ instead of $\exists^\ell_\epsilon$?
In affine logic, we need it to ensure that the first Weihrauch program halts:

$$\forall^\ell x^1(A^\bullet(x) \multimap \exists^\ell y^1 B^\bullet(x,y))$$

$$\multimap$$

$$\forall^\ell u^1(C^\bullet(u) \multimap \exists^\ell v^1 D^\bullet(u,v))$$

Here, an affine proof might drop the premise.
Thus, it does not (necessarily) contain a method for producing x with $A(x)$ from u with $C(u)$.
Conclusion: Affine logic prevents us from using the premise more than once, but not from using the premise not at all.

Making the result more pretty

How did we solve this problem?
What happens if we use $\exists^\ell_\epsilon$ instead of $\exists^\ell$?

$$\forall^\ell x^1(A^\bullet(x) \multimap \exists y^1(\epsilon =_0 0 \otimes B^\bullet(x,y)))$$

$$\multimap$$

$$\forall^\ell u^1(C^\bullet(u) \multimap \exists v^1(\epsilon =_0 0 \otimes D^\bullet(u,v)))$$

Assume there were an affine proof that does not use the premise.
C must not contain the variable $\epsilon \quad \Longrightarrow \quad C \to \bot$
This entails a trivial Weihrauch reduction

This solution is a bit “hacky”, can it be improved?
Yes, but not in affine logic

Making the result more pretty

Why do we care for affine logic?
Our verifying system $\text{E-PA}^\omega + \text{QF-AC}^{0,0} + \Gamma$ is classical

$$\frac{\vdash \Gamma}{\vdash \Gamma, |A|^0_v}\,(\text{w})$$

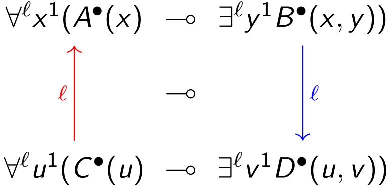
In a classical verifying system, interpreting weakening is trivial.

Linear Dialectica does not retrieve more information than Affine Dialectica

Solution: Use something that is not Dialectica in order to capture that Linear Logic has no (affine) weakening.

Idea: Apply tags to linear predicates

Suppose we have a proof



How can we make sure that the proof is structured in a certain way?

Idea: Apply tags to both *negatively occurring* linear predicates. Follow these tags through the proof. If both left ℓ and both right ℓ have the same tag, this implies a proof in the style of a Weihrauch reduction.
Next step: Show that in a linear setting, this is the *only* possible configuration for tags.

Simplified phase semantics for Linear Logic

Phase space

Multiplicative monoid $P := \{0, 1\}$ together with antiphases $\perp := \{1\} \subseteq P$.

Involution

$Q^{\perp} := \{p \in P : \forall q \in Q \quad pq \in \perp\}$ for $Q \subseteq P$

Facts

Subsets Q of P with $Q^{\perp\perp} = Q$. Q is *valid* iff $1 \in Q$

- $0 := \emptyset$: Non-valid fact
- $1 := \{1\}$: Valid fact
- $\top := \{0, 1\}$: Valid fact

$\{0\}^{\perp\perp} = 0^{\perp} = \top \neq \{0\} \quad \Longrightarrow \quad \{0\}$ is not a fact

Simplified phase semantics for Linear Logic

Assume that Q and R are subsets of the phase space P

Connectives

$$
\begin{aligned}
Q \otimes R &:= \{qr : q \in Q \text{ and } r \in R\} \\
Q \,\&\, R &:= Q \cup R \\
?Q &:= Q \cup 1
\end{aligned}
$$

$$
\begin{aligned}
Q \parr R &:= (Q^{\perp} \otimes Q^{\perp})^{\perp} \\
Q \oplus R &:= Q \cap R \\
!Q &:= Q \cap 1
\end{aligned}
$$

$$P \multimap Q = \{s : qs \in R \text{ for all } q \in Q\}$$

Simplified phase semantics for Linear Logic

The following is valid

$$1$$

since the fact 1 is valid.

Is the following valid?

$$\top \multimap 1$$

We know $\top \multimap 1 = 0$ is not a valid fact.

Conclusion: Our semantics reject (affine) weakening!

Soundness of phase semantics

Lemma

Let Γ *be a set of formulas such that* $\text{E-PA}^\omega + \Gamma + \text{QF-AC}^{0,0}$ *is consistent.*
If $\text{E-LPA}^\omega_\ell + \Gamma^\bullet$ *proves the sequent*

$$\vdash \Delta,$$

then it holds semantically with respect to P*, i.e.*

$$\Vdash \Delta.$$

Corollary

$\text{E-LPA}^\omega_\ell + \Gamma^\bullet$ *rejects (affine) weakening for* Γ *where* $\text{E-PA}^\omega + \Gamma + \text{QF-AC}^{0,0}$ *is consistent.*

Dialectica with tags

We introduce a new modified Dialectica that applies one of two possible tags to each linear predicate.

Tags of linear predicates that occur

- negatively can be chosen arbitrarily,
- positively are determined by the functional interpretation.

For simplification, we use tags with the following colors:

- red tags with semantics 1,
- blue tags with semantics 0 or $\top$.

In the case of blue tags, the choice must be uniform.

Proving the (prettier) Theorem

We apply the following tags (red and blue):

$$\vdash \forall^\ell x^1(A^\bullet(x) \multimap \exists^\ell y^1 B^\bullet(x,y)) \multimap \forall^\ell u^1(C^\bullet(u) \multimap \exists^\ell v^1 D^\bullet(u,v)).$$

The functional interpretation might give one of the following colorings:

$$\vdash \forall^\ell x^1(A^\bullet(x) \multimap \exists^\ell y^1 B^\bullet(x,y)) \multimap \forall^\ell u^1(C^\bullet(u) \multimap \exists^\ell v^1 D^\bullet(u,v)).$$

$$\vdash \forall^\ell x^1(A^\bullet(x) \multimap \exists^\ell y^1 B^\bullet(x,y)) \multimap \forall^\ell u^1(C^\bullet(u) \multimap \exists^\ell v^1 D^\bullet(u,v)).$$

$$\vdash \forall^\ell x^1(A^\bullet(x) \multimap \exists^\ell y^1 B^\bullet(x,y)) \multimap \forall^\ell u^1(C^\bullet(u) \multimap \exists^\ell v^1 D^\bullet(u,v)).$$

$$\vdash \forall^\ell x^1(A^\bullet(x) \multimap \exists^\ell y^1 B^\bullet(x,y)) \multimap \forall^\ell u^1(C^\bullet(u) \multimap \exists^\ell v^1 D^\bullet(u,v)).$$

Only the second variant is possible for both semantics of blue tags.

Proving the (prettier) Theorem

$$\vdash \forall^{\ell} x^1 (A^{\bullet}(x) \multimap \exists^{\ell} y^1 B^{\bullet}(x,y)) \multimap \forall^{\ell} u^1 (C^{\bullet}(u) \multimap \exists^{\ell} v^1 D^{\bullet}(u,v)).$$

The linear predicates with blue tags may be replaced by a certain class of formulas.
We choose:

$$\ell(x) :\equiv \ell(x) \otimes (\epsilon =_0 0)$$

Thus, we can use the above sequent to prove the following:

$$\vdash \forall^{\ell} x^1 (A^{\bullet}(x) \multimap \exists^{\ell}_{\epsilon} y^1 B^{\bullet}(x,y)) \multimap \forall^{\ell} u^1 (C^{\bullet}(u) \multimap \exists^{\ell}_{\epsilon} v^1 D^{\bullet}(u,v)).$$

In fact, the provability of both sequents in E-LPA$^{\omega}_{\ell}$ $+\Gamma^{\bullet}$ is equivalent. $\square$

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$$C(u) \to t \cdot u{\downarrow} \wedge A(t \cdot u)$$

and

$$C(u) \wedge B(t \cdot u, y) \to s \cdot j(u, y){\downarrow} \wedge D(u, s \cdot j(u, y))$$

for some closed terms t^1 *and* s^1 *of* $\mathcal{L}(\text{E-PA}^\omega)$*.*

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