Reverse mathematics of combinatorial principles over a weak base theory

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(part of joint project with
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Reverse mathematics

- Reverse mathematics studies the strength of axioms needed to prove various mathematical theorems. This is done by proving implications between the theorems and/or various logical principles over a relatively weak base theory.
- Often, the theorem is Π^1_2 of the form $\forall X \exists Y \psi$, and its strength is related to the difficulty of computing Y given X.
- In the early days, many theorems were proved equivalent to one of a few principles like "for each set, its jump exists" etc.
- Later work: theorems from e.g. Ramsey theory form a great mess of (non)implications (the "reverse mathematics zoo").
- Today's talk: we focus *on the base theory*.

Usual base theory: RCA_0

Language:

vbles $x, y, z, \ldots, i, j, k \ldots$ for natural numbers;
vbles $X, Y, Z, \ldots$ for sets of naturals; symbols $+, \cdot, 2^x, \leq, 0, 1, \in$.

Axioms:

- $+, \cdot, 2^x$ etc. have their usual basic properties,
- Δ^0_1 comprehension: if $\bar{X} = X_1, \ldots, X_k$ are sets and $\psi(x, \bar{X})$ is computable relative to $\bar{X}$, then $\{n : \psi(n, \bar{X})\}$ is a set.
- Σ^0_1 induction: if $\bar{X}$ are sets and $\psi(x, \bar{X})$ is r.e. relative to $\bar{X}$, then $\psi(0, \bar{X}) \wedge \forall n \left(\psi(n, \bar{X}) \Rightarrow \psi(n+1, \bar{X})\right) \Rightarrow \forall n \, \psi(n, \bar{X})$.

Weaker base theory: RCA_0^*

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Introduced in Simpson-Smith 86. Studied a bit both in traditional reverse maths and "reverse recursion theory". Most results have had the form "this still holds over RCA_0^*" or "this is equivalent to RCA_0".

What is the zoo like over RCA_0^*?

Main principles we consider:

- RT^2_2: for every $f\colon [\mathbb{N}]^2 \to 2$ there is infinite $H \subseteq \mathbb{N}$ such that $f{\restriction}_{[H]^2} = \text{const}$.
- CAC: for every partial ordering $\preccurlyeq$ on $\mathbb{N}$ there is infinite $H \subseteq \mathbb{N}$ such that $(H, \preccurlyeq)$ is either a chain or an antichain.
- ADS: in every linear ordering $\preccurlyeq$ on $\mathbb{N}$ there is either an infinite ascending sequence or an infinite descending sequence.
- CRT^2_2: for every $f\colon [\mathbb{N}]^2 \to 2$ there is infinite $H \subseteq \mathbb{N}$ such that $\forall x \in H\, \exists y \in H\, \forall z \in H\, \big(z \geq y \Rightarrow \big(f(x,z) = f(x,y)\big)\big)$.

Over RCA_0, we have $\mathrm{RT}^2_2 \Rightarrow \mathrm{CAC} \Rightarrow \mathrm{ADS} \Rightarrow \mathrm{CRT}^2_2$. (HS07; LST 13)
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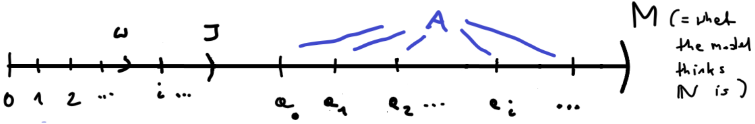
What does failure of $I\Sigma^0_1$ mean?

If a model $(M, \mathcal{X})$ of RCA_0^* is not a model of RCA_0, then Σ^0_1 induction fails:

- there is a Σ^0_1 definable *proper cut* J (contains 0, closed downwards and under $+1$),
- there is an infinite (=unbounded) set $A \in \mathcal{X}$ s.t. $A = \{a_i : i \in J\}$ enumerated in increasing order. We can say that $|A| = J$.



Two flavours of Ramsey-theoretic principles

In RCA_0^*, "for every f there exists infinite $H \subseteq \mathbb{N}$..."
can mean (at least) one of two things:

- "for every f there exists unbounded $H \subseteq \mathbb{N}$..." (*normal* version),
- "for every f there exists $H \subseteq \mathbb{N}$ with $|H| = \mathbb{N}$ s.t. ..." (*fat* version).

We will consider both versions, starting with normal.

Normal versions: relativization to cuts

Given a proper cut J in $(M, \mathscr{X}) \models \mathrm{RCA}_0^*$, the family $\mathrm{Cod}(M/J)$ is $\{B \cap J : B \in \mathscr{X}\}$. This family depends only on M and J, not on $\mathscr{X}$.

If J is closed under $x \mapsto 2^x$, then $(J, \mathrm{Cod}(M/J))$ satisfies WKL_0^* ($= \mathrm{RCA}_0^*$ + "every infinite tree in $\{0,1\}^{\mathbb{N}}$ has a length-$\mathbb{N}$ path").

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We have a more general sufficient condition for this equivalence. Note that l.h.s. does not depend on J, r.h.s does not depend on $\mathscr{X}$!

A useful fact about coding

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We will prove the $(\Rightarrow)$ direction for RT_2^2.
Both directions are similar and rely on the following fact about Cod.

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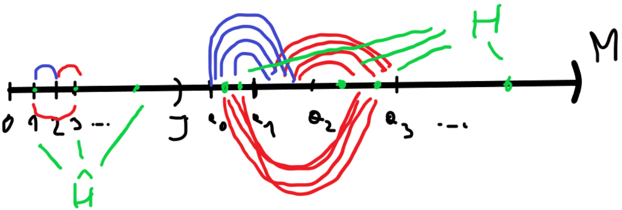
Fact (essentially Chong-Mourad 1990)

Let $(M,\mathscr{X}) \models \mathrm{RCA}_0^*$*, let* J *be a proper cut in* M*, let* $\mathscr{X} \ni A = \{a_i : i \in J\}$*.*
Then for every $\mathscr{X} \ni B \subseteq A$*, the set* $\{i \in J : a_i \in B\}$ *is in* $\mathrm{Cod}(M/J)$*.*

Proving $(M, \mathscr{X}) \models \mathrm{RT}^2_2 \Rightarrow (J, \mathrm{Cod}(M/J)) \models \mathrm{RT}^2_2$.

Proof.

Let $A \in \mathscr{X}$ be such that $A = \{a_i : i \in J\}$. Let $f : [J]^2 \to 2$ be coded.
Define a colouring of $[A]^2$ by $\check{f}(a_{i_1}, a_{i_2}) = f(i_1, i_2)$
Extend $\check{f}$ to $[M]^2$ by looking at nearest elements of A.
Use RT^2_2 in $(M, \mathscr{X})$ to get $H \subseteq M$ homogeneous for $\check{f}$.



By Chong-Mourad, $\widehat{H} = \{i \in J : H \cap (a_{i-1}, a_i] \neq \emptyset\}$ is in $\mathrm{Cod}(M/J)$.
This set $\widehat{H}$ is homogeneous for f. □

Normal versions: what else can be said

If Ψ is the normal version of a Ramsey-theoretic principle (such as one of our RT^2_2, CAC, ADS, CRT^2_2), the following things follow from the characterization in terms of cuts:

- If $(M, \mathscr{X}) \models \Psi$ and (lightface) Σ_1 induction fails in M, then $(M, \Delta_1\text{-Def}(M)) \models \Psi$. I.e., Ψ is computably true in M!
- $\mathrm{RCA}^*_0 + \Psi$ does not prove any Π^0_3 sentences that are unprovable in RCA^*_0 (i.e., $\mathrm{RCA}^*_0 + \Psi$ is Π^0_3-conservative over RCA^*_0).
- $\mathrm{RCA}^*_0 + \Psi$ is arithmetically conservative over RCA^*_0 iff $\mathrm{WKL}^*_0 \vdash \Psi$ (and then we also have Π^1_1-conservativity).
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Also worth mentioning:

- The implications $\mathrm{RT}^2_2 \Rightarrow \mathrm{CAC} \Rightarrow \mathrm{ADS}$ and $\mathrm{RT}^2_2 \Rightarrow \mathrm{CRT}^2_2$ still hold in RCA^*_0. We do not know if $\mathrm{CAC} \Rightarrow \mathrm{CRT}^2_2$ holds.

Fat versions: what is ADS?

Many principles have one natural fat version. In many cases it is easily seen to imply RCA_0. (E.g. fat-RT^2_2, by Yokoyama 2013.)

For ADS, the issue is delicate:

- fat-$\mathrm{ADS}^{\mathrm{set}}$: "for every linear ordering $\preccurlyeq$ on $\mathbb{N}$, there is H with $|H| = \mathbb{N}$ s.t. $\preccurlyeq, \leq$ either always agree or always disagree on H".
- fat-$\mathrm{ADS}^{\mathrm{seq}}$: "for every linear ordering $\preccurlyeq$ on $\mathbb{N}$, there is $h\colon \mathbb{N} \to \mathbb{N}$ which is either an ascending or a descending sequence in $\preccurlyeq$".

RCA^*_0 proves fat-$\mathrm{RT}^2_2 \Rightarrow$ fat-CAC $\Rightarrow$ fat-$\mathrm{ADS}^{\mathrm{set}} \Rightarrow$ fat-$\mathrm{ADS}^{\mathrm{seq}}$.
Over RCA_0, we also have fat-$\mathrm{ADS}^{\mathrm{set}} \Leftrightarrow$ fat-$\mathrm{ADS}^{\mathrm{seq}}$.

Some fat principles are strong

Theorem

Over RCA_0^*, fat-$\mathrm{ADS}^{\mathrm{set}}$ *implies* RCA_0.

Proof.

- Assume $\mathrm{I}\Sigma_1^0$ fails, so we have unbounded $A = \{a_i : i \in J\}$ for proper Σ_1^0-definable cut J.
- If $x \in [a_i, a_{i+1})$ and $y \in [a_j, a_{j+1})$ for $i < j$, set $x \preccurlyeq y$.
- If $x, y \in [a_i, a_{i+1})$, set $x \preccurlyeq y$ iff $x > y$.
- Then all $\preccurlyeq$-ascending sets have cardinality at most J, and all $\preccurlyeq$-descending sets are finite. So, fat-$\mathrm{ADS}^{\mathrm{set}}$ fails.

□

Some fat principles are less strong

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Over RCA_0^**, (normal)* ADS *and* fat-$\mathrm{ADS}^{\mathrm{seq}}$ *are equivalent.*

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Over RCA_0^*, *(normal)* ADS *and* fat-$\mathrm{ADS}^{\mathrm{seq}}$ *are equivalent.*

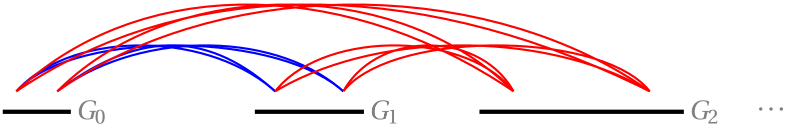
We will prove the implication in WKL_0^*, using a variant of the *grouping principle* (cf. Patey-Yokoyama 2018) specific to RCA_0^*.

Definition

EFG (*Ever fatter grouping* principle) says: “for every $f\colon [\mathbb{N}]^2 \to 2$, there exists an infinite family of finite sets $G_0 < G_1 < \ldots$ such that:

- the cardinalities $|G_i|$ grow to $\mathbb{N}$ as i increases,
- for each $i < j$, we have $f\restriction_{G_i \times G_j} = \mathrm{const}$”.

EFG pictured



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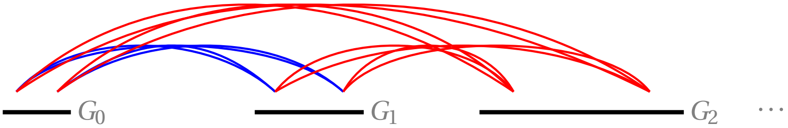


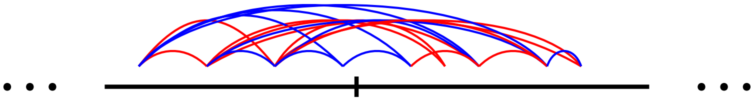
Theorem

$\mathrm{WKL}_0^* + \neg \mathrm{I}\Sigma_1^0$ *proves* EFG.

Proving EFG

We use the “thinning out from below/from above” method (cf. K-Yokoyama 2020). Let $f\colon [\mathbb{N}]^2 \to 2$ be given.

- Assuming $\neg \mathrm{I}\Sigma^0_1$, we have the usual cut J and set $A = \{a_i : i \in J\}$.
- W.l.o.g., we have (i) $|[a_i, a_{i+1})| \gg 2^{|[0,a_i)|}$ and (ii) $|[a_i, a_{i+1})| \gg |J|$. Let $G^0_0 = [0, a_0), G^0_1 = [a_0, a_1), G^0_2 = [a_1, a_2)$ etc.



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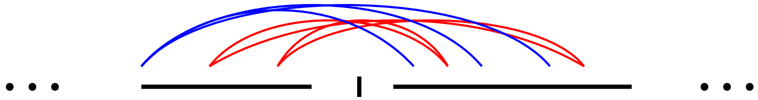
- Using (i), take large $G^1_0 \subseteq G^0_0, G^1_1 \subseteq G^0_1, G^1_2 \subseteq G^0_2, \dots$ so that $f\restriction_{\{x\}\times G^1_j}$ constant for each $x \in G^0_i$, $i < j$.



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- Given fixed k, using (ii) lets us take large $G^2_k \subseteq G^1_k, \ldots, G^2_0 \subseteq G^1_0$ so that $f{\restriction}_{G^2_i \times G^2_j}$ constant for each $i < j \leq k$.

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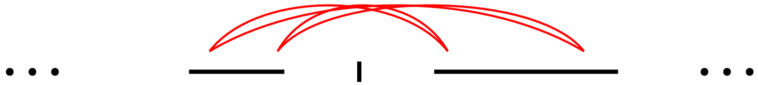
- Such finite approximations to a witness to EFG form a binary tree. Take infinite path provided by WKL. □



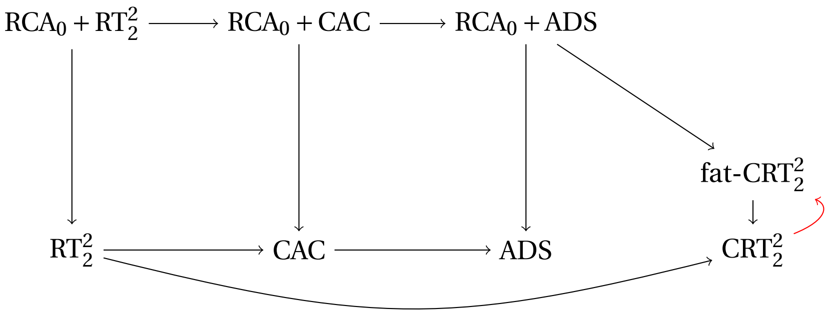
Proving ADS + EFG $\Rightarrow$ fat-ADS$^{\text{seq}}$

Let linear ordering $\preccurlyeq$ on $\mathbb{N}$ be given. We can assume $\neg I\Sigma^0_1$.

- By EFG, we get $G_0 < G_1 < G_2 \ldots$ s.t. for $i \neq j$, points in G_i are either all $\preccurlyeq$-above or all $\preccurlyeq$-below all points in G_j.
- So, there is an induced $\preccurlyeq$-ordering on the set of G_i's. By ADS, there is (w.l.o.g.) a descending sequence $G_{i_0} \succcurlyeq G_{i_1} \succcurlyeq G_{i_2} \ldots$ where $i_0 < i_1 < i_2 \ldots$ The numbers $|G_{i_k}|$ grow to $\mathbb{N}$ with k.
- Build length-$\mathbb{N}$ $\preccurlyeq$-decreasing sequence by enumerating G_{i_0} in $\preccurlyeq$-decreasing order, then G_{i_1} in $\preccurlyeq$-decreasing order etc. □

A similar argument shows that $WKL^*_0 + CRT^2_2$ proves fat-CRT^2_2.
For colourings given by linear orderings (= transitive colourings), WKL can be eliminated from proof of EFG.

Normal and fat principles: summary



Red implication known in the presence of WKL.

The curious case of COH

The principle COH is: "for every family $\{R_x : x \in \mathbb{N}\}$ of subsets of $\mathbb{N}$, there exists infinite $H \subseteq \mathbb{N}$ such that for each x, either $\forall^\infty z \in H\,(z \in R_x)$ or $\forall^\infty z \in H\,(z \notin R_x)$".

This strengthens CRT^2_2: think of $f(x,y)$ as $y \in R_x$.
But here, $f(x,\cdot)$ must stabilize on H for each x, not just for $x \in H$.

Over RCA_0, RT^2_2 proves COH. Even if we do not require $|H| = \mathbb{N}$, COH has a certain "fat" aspect, due to the "for each x" condition.

How strong is COH over RCA^*_0?

The curious case of COH (cont'd)

Theorem

A model of RCA_0^* *of the form* $(M, \Delta_1\text{-Def}(M))$ *never satisfies* COH.

Corollary

$\mathrm{RCA}_0^* + \mathrm{RT}_2^2$ *does not prove* COH.

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Proof of Theorem.

Like over RCA_0, COH implies that for any set A and any two disjoint $\Sigma_2(A)$-sets, there is a set B and a $\Delta_2(B)$-set separating them.
But RCA_0^* is enough to prove that there are disjoint Σ_2-sets with no separating (lightface) Δ_2-set. □

That is pretty much all we know about COH over RCA_0^*.

Some open problems

- Does ADS or CAC imply CRT^2_2 over RCA^*_0?
- Can the grouping principle EFG be proved in $\mathsf{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$?
- What is the strength of COH?
  Does it imply $\mathrm{I}\Sigma^0_1$? Is it Π^0_3-conservative over RCA^*_0?

Coming soon...

Small teaser: things we have just started writing up.

- For Ψ a Π^1_2 statement, $\mathrm{RCA}^*_0 + \Psi$ is Π^1_1-conservative over $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ iff $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ proves Ψ.
- The above is false without the extra condition $\neg\mathrm{I}\Sigma^0_1$.
- For any n, the maximal Π^1_2 theory that is Π^1_1-conservative over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ is recursively axiomatized.
  (Here $\mathrm{B}\Sigma^0_n$ is basically another name for $\mathrm{I}\Delta^0_n$.)
- If $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ is $\forall\Pi^0_5$-conservative over $\mathrm{B}\Sigma^0_2$, then it is Π^1_1-conservative over $\mathrm{B}\Sigma^0_2$.
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Coming soon...

Small teaser: things we have just started writing up.

- For Ψ a Π^1_2 statement, $\mathrm{RCA}^*_0 + \Psi$ is Π^1_1-conservative over $\mathrm{RCA}^*_0 + \neg\mathrm{I}\Sigma^0_1$ iff $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ proves Ψ.
- The above is false without the extra condition $\neg\mathrm{I}\Sigma^0_1$.
- For any n, the maximal Π^1_2 theory that is Π^1_1-conservative over $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ is recursively axiomatized.
  (Here $\mathrm{B}\Sigma^0_n$ is basically another name for $\mathrm{I}\Delta^0_n$.)
- If $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ is $\forall\Pi^0_5$-conservative over $\mathrm{B}\Sigma^0_2$, then it is Π^1_1-conservative over $\mathrm{B}\Sigma^0_2$.
- ...

Thank you!