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On the descriptive complexity of Fourier dimension and Salem sets.

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Motivation

Hausdorff dimension is a standard notion in geometric measure theory. It estimates the “size” of a set by means of open covers of it.

The effective Hausdorff dimension can be defined in terms of effectively presented open covers (or in terms of martingales).

If we restrict our attention to singletons, we can characterize the effective Hausdorff dimension of $\{\xi\}$ by means of the Kolmogorov complexity of ξ (J. H. Lutz and Mayordomo 2008).

Motivation

Recently Jack and Neil Lutz proved a point-to-set principle linking the (classical) Hausdorff dimension of a set with the (relative) effective Hausdorff dimension of its points.

Theorem ((J. H. Lutz and N. Lutz 2017))

For every $E \subset \mathbb{R}^d$

$$\dim_{\mathcal{H}}(E) = \min_{A \subset \mathbb{N}} \sup_{x \in E} \dim_{\mathcal{H}}^{A}(x)$$

Can we have something similar for the Fourier dimension?

Hausdorff dimension

The formal definition of the Hausdorff dimension is a bit involved.

For $A \in \mathcal{B}(\mathbb{R}^d)$ we can characterize the Hausdorff dimension by

$$\dim_{\mathcal{H}}(A) = \sup\{s : (\exists \mu \in \mathbb{P}(A))\,(\exists c > 0)\,(\forall x \in \mathbb{R}^d)\,(\forall r > 0)\,(\mu(B(x,r)) \leq cr^s)\}$$

This is a consequence of Frostman's lemma.

Fourier dimension

Let μ be a finite Borel measure.

Fourier transform of μ: $\widehat{\mu}\colon \mathbb{R}^d \to \mathbb{C}$ defined as

$$\widehat{\mu}(\xi) := \int e^{-i\xi\cdot x}\, d\mu(x)$$

The **Fourier dimension** of $A \subset \mathbb{R}^d$ is defined as

$$\dim_{\mathrm{F}}(A) := \sup\{s \in [0, d] \;:\; (\exists \mu \in \mathbb{P}(A))\,(\exists c > 0)\,(\forall x \in \mathbb{R}^d)\,(|\widehat{\mu}(x)| \le c\,|x|^{-s/2})\}$$

Fourier dimension

Proposition (Folklore)

For every $A \in \mathcal{B}(\mathbb{R}^d)$ *we have:* $\dim_{\mathrm{F}}(A) \leq \dim_{\mathcal{H}}(A)$

A set A s.t. $\dim_{\mathcal{H}}(A) = \dim_{\mathrm{F}}(A)$ is called **Salem set**.

$\emptyset$, $[0,1]$ are Salem subsets of $[0,1]$.
Cantor middle-third set has $\dim_{\mathcal{H}} = \dfrac{\log(2)}{\log(3)}$ but $\dim_{\mathrm{F}} = 0$.

Deterministic (non-trivial) Salem sets are rare.

Jarnìk's fractal

One of the first explicit examples of (non-trivial) Salem sets.

For $\alpha \geq 0$, $E(\alpha)$ is the set of α-well approximable numbers.

$$E(\alpha) := \left\{ x \in [0,1] \, : \, (\exists^\infty n, m) \left(\left| x - \frac{n}{m} \right| \leq \frac{1}{m^{2+\alpha}} \right) \right\}$$

It is Salem with dimension $\frac{2}{2+\alpha}$. For $\alpha = 0$ we have $E(\alpha) = [0,1]$ by Dirichlet's theorem.

Jarnìk's fractal is a $\mathbf{\Pi}^0_3$ set. However we can prove that there is a closed subset $S(\alpha)$ of $E(\alpha)$ which is Salem of the same dimension.

$S(\alpha)$ can be written as $\displaystyle\bigcap_{k \in \mathbb{N}} G_k = \bigcap_{k \in \mathbb{N}} \bigcup_{i \leq N_k} I_i$

Question

During the IMS Graduate Summer School in Logic in 2018, Slaman asked:

"What is the descriptive complexity of the family of closed Salem sets in $[0, 1]$?"

Descriptive complexity

We will locate the family of closed Salem sets in the Borel hierarchy

Let X and Y be Polish spaces.

$A \subset X$ is **Wadge reducible** to $B \subset Y$ ($A \leq_W B$) if there is a continuous map $f \colon X \to Y$ s.t.

$$x \in A \iff f(x) \in B$$

B is called $\mathbf{\Gamma}$-hard if, for every $A \in \mathbf{\Gamma}(2^{\mathbb{N}})$, $A \leq_W B$.
B is called $\mathbf{\Gamma}$-complete if it is $\mathbf{\Gamma}$-hard and $B \in \mathbf{\Gamma}(Y)$.

Salem sets in $[0, 1]$

We work in the hyperspace $\mathbf{K}([0,1])$ of compact subsets of $[0,1]$ endowed with the Vietoris topology.

Lemma (Marcone, Reimann, Slaman, V.)

- $\{(A,p) \in \mathbf{K}([0,1]) \times [0,1) : \dim_{\mathcal{H}}(A) > p\}$ *is* $\mathbf{\Sigma}^0_2$;
- $\{(A,p) \in \mathbf{K}([0,1]) \times (0,1] : \dim_{\mathcal{H}}(A) \geq p\}$ *is* $\mathbf{\Pi}^0_3$

- $\{(A,p) \in \mathbf{K}([0,1]) \times [0,1) : \dim_{\mathrm{F}}(A) > p\}$ *is* $\mathbf{\Sigma}^0_2$;
- $\{(A,p) \in \mathbf{K}([0,1]) \times (0,1] : \dim_{\mathrm{F}}(A) \geq p\}$ *is* $\mathbf{\Pi}^0_3$

Salem sets in $[0, 1]$

Proof (Sketch).

We prove the statement for the Hausdorff dimension.

Salem sets in [0, 1]

Proof (Sketch).

We prove the statement for the Hausdorff dimension. By Frostman's lemma, $\dim_{\mathcal{H}}(A)$ is the sup of

$$\{s : (\exists \mu \in \mathbb{P}(A))(\exists c > 0)(\forall x \in \mathbb{R})(\forall r > 0)(\mu(B(x, r)) \leq cr^s)\}$$

□

Salem sets in $[0, 1]$

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using (Kechris 1995)

□

Salem sets in [0, 1]

Proof (Sketch).

We prove the statement for the Hausdorff dimension. By Frostman's lemma, $\dim_{\mathcal{H}}(A)$ is the sup of

$$\{s : (\exists \mu \in \mathbb{P}(A))(\exists c > 0) \underbrace{(\forall x \in \mathbb{R})(\forall r > 0)(\underbrace{\mu(B(x,r)) \leq cr^s}_{\mathbf{\Pi}^0_1})}_{\mathbf{\Pi}^0_1}\}$$

intersection of closed sets

□

Salem sets in $[0,1]$

Proof (Sketch).

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$$\{s : \underbrace{(\exists \mu \in \mathbb{P}(A))\,(\exists c > 0)\,\underbrace{(\forall x \in \mathbb{R})(\forall r > 0)(\underbrace{\mu(B(x,r)) \leq c r^s}_{\mathbf{\Pi}^0_1})}_{\mathbf{\Pi}^0_1}}_{\mathbf{\Sigma}^0_2}\}$$

we can quantify over rationals

□

Salem sets in $[0, 1]$

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$$\{s : \underbrace{(\exists\mu\in\mathbb{P}(A))\,\underbrace{(\exists c>0)\,\underbrace{(\forall x\in\mathbb{R})(\forall r>0)(\underbrace{\mu(B(x,r))\le cr^s}_{\boldsymbol{\Pi}^0_1})}_{\boldsymbol{\Pi}^0_1}}_{\boldsymbol{\Sigma}^0_2}}_{\boldsymbol{\Sigma}^0_2}\}$$

projection of a $\boldsymbol{\Sigma}^0_2$ along a metrizable compact space is $\boldsymbol{\Sigma}^0_2$ (Andretta and Marcone 1997)

□

Salem sets in $[0, 1]$

Proof (Sketch).

We prove the statement for the Hausdorff dimension. By Frostman's lemma, $\dim_{\mathcal{H}}(A)$ is the sup of

a $\mathbf{\Sigma}_2^0$ set $D(A)$.

□

Salem sets in $[0, 1]$

Proof (Sketch).

We prove the statement for the Hausdorff dimension. By Frostman's lemma, $\dim_{\mathcal{H}}(A)$ is the sup of

a $\mathbf{\Sigma}^0_2$ set $D(A)$.

Hence

$$\dim_{\mathcal{H}}(A) > p \Leftrightarrow \sup D(A) > p \Leftrightarrow (\exists q > p)(q \in D(A)) \quad \text{is } \mathbf{\Sigma}^0_2$$

$$\dim_{\mathcal{H}}(A) \geq p \Leftrightarrow \sup D(A) \geq p \Leftrightarrow (\forall q < p)(q \in D(A)) \quad \text{is } \mathbf{\Pi}^0_3$$

□

Salem sets in $[0, 1]$

Theorem (Marcone, Reimann, Slaman, V.)

The set of closed Salem subsets of $[0,1]$ *is* $\mathbf{\Pi}^0_3$.

Proof.

For Borel sets we have $\dim_{\mathrm{F}}(A) \leq \dim_{\mathcal{H}}(A)$.

Being a Salem set can be written as

$$(\forall r \in \mathbb{Q})\ (\dim_{\mathcal{H}}(A) > r \rightarrow \dim_{\mathrm{F}}(A) > r)$$

□

Salem sets in $[0, 1]$

Lemma (Marcone, Reimann, Slaman, V.)

For every $p \in [0,1]$ *there is a continuous (in fact computable) map* $f_p \colon 2^{\mathbb{N}} \to \mathscr{S}([0,1])$ *s.t.*

$$\dim(f_p(x)) = \begin{cases} p & \text{if } x \in Q_2 \\ 0 & \text{if } x \notin Q_2 \end{cases}$$

where $Q_2 = \{x \in 2^{\mathbb{N}} : (\forall^\infty n)(x(n) = 0)\}$ *is* $\mathbf{\Sigma}^0_2$*-complete.*

Idea of the proof: each time we see a 1 we "control" the Hausdorff (and hence Fourier) dimension, so that if there are infinitely many 1's then the set will have 0 Hausdorff (and hence Fourier) dimension.

Salem sets in [0, 1]

Theorem (Marcone, Reimann, Slaman, V.)

The sets

$$\{(A,p) \in \mathbf{K}([0,1]) \times [0,1) \ : \ \dim_{\mathcal{H}}(A) > p\},$$
$$\{(A,p) \in \mathbf{K}([0,1]) \times [0,1) \ : \ \dim_{\mathrm{F}}(A) > p\},$$

are $\mathbf{\Sigma}^0_2$*-complete.*

Salem sets in $[0, 1]$

Theorem (Marcone, Reimann, Slaman, V.)

The sets

$$\{(A, p) \in \mathbf{K}([0,1]) \times (0,1] \,:\, \dim_{\mathcal{H}}(A) \geq p\},$$
$$\{(A, p) \in \mathbf{K}([0,1]) \times (0,1] \,:\, \dim_{\mathrm{F}}(A) \geq p\},$$
$$\{A \in \mathbf{K}([0,1]) \,:\, A \in \mathscr{S}([0,1])\}$$

are $\mathbf{\Pi}^0_3$*-complete.*

Salem sets in [0, 1]

Proof (Sketch):

$$P_3 := \{x \in 2^{\mathbb{N}\times\mathbb{N}} : (\forall m)(\forall^\infty n)(x(m,n) = 0)\} \qquad \mathbf{\Pi}^0_3\text{-complete}$$

Given x, modify it so that if a row has infinitely many 1s then so do all the next ones.

Consider countably many disjoint closed intervals $\{T_n\}_n \subset [0,1]$.

We build a fractal within T_0 with $\dim_{\mathcal{H}} = 1$ and $\dim_{\mathrm{F}} = 0$. We then build a Salem set S_{i+1} within T_{i+1} according to the i-th row of x, so that

$$\dim(S_{i+1}) = \begin{cases} 1 - 2^{i+1} & \text{if } x(i,\cdot) \text{ has finitely many 1} \\ 0 & \text{otherwise} \end{cases}$$

□

How about closed subset of $[0,1]^d$?

The upper bounds on the complexities are the same (the proof is based on the compactness of the ambient space).

Problem: the Fourier dimension is sensitive to the ambient space.

If a set A is contained in a m-dimensional hyperplane (with $m < d$) then $\dim_{\mathrm{F}}(A) = 0$.

Some “curvature” is necessary to have positive Fourier dimension.

Solutions?

We can exploit a “higher-dimensional analogue” of Jarník’s fractal, recently defined by Fraser and Hambrook.

Theorem ((Fraser and Hambrook 2019))

For every $\alpha \geq 0$, the set $E(K, B, \alpha)$ is a Salem set of dimension $2d/(2+\alpha)$.

Salem sets in $[0,1]^d$

Similarly to the one-dimensional case we have:

Lemma (Marcone, V.)

For every $p \in [0,d]$ there exists a continuous (in fact computable) map $f_p \colon 2^{\mathbb{N}} \to \mathscr{S}([0,1]^d)$ s.t.

$$\dim(f_p(x)) = \begin{cases} p & \textit{if } x \in Q_2 \\ 0 & \textit{if } x \notin Q_2 \end{cases}$$

where $Q_2 = \{x \in 2^{\mathbb{N}} : (\forall^\infty n)(x(n) = 0)\}$ is $\mathbf{\Sigma}^0_2$-complete.

Salem sets in $[0,1]^d$

Theorem (Marcone, V.)

For every $d \geq 1$, the sets

$$\{(A,p) \in \mathbf{K}([0,1]^d) \times [0,d) \,:\, \dim_{\mathcal{H}}(A) > p\},$$
$$\{(A,p) \in \mathbf{K}([0,1]^d) \times [0,d) \,:\, \dim_{\mathrm{F}}(A) > p\}$$

are $\mathbf{\Sigma}^0_2$-complete. Moreover the sets

$$\{(A,p) \in \mathbf{K}([0,1]^d) \times (0,d] \,:\, \dim_{\mathcal{H}}(A) \geq p\},$$
$$\{(A,p) \in \mathbf{K}([0,1]^d) \times (0,d] \,:\, \dim_{\mathrm{F}}(A) \geq p\},$$
$$\{A \in \mathbf{K}([0,1]^d) \,:\, A \in \mathscr{S}([0,1]^d)\}$$

are $\mathbf{\Pi}^0_3$-complete.

Relaxing compactness

Do things change if we move to $\mathbb{R}^d$?

Both Hausdorff and Fourier dimensions are preserved when moving from $[0,1]^d$ to $\mathbb{R}^d$.

In particular, every Salem set of $[0,1]^d$ is still Salem when seen as a subset of $\mathbb{R}^d$.

Hardness results (lower bounds) are corollaries, while upper bounds are more delicate.

Topology on $\mathbf{F}(\mathbb{R}^d)$

There is no "canonical" topology on the hyperspace $\mathbf{F}(\mathbb{R}^d)$ of closed subsets of $\mathbb{R}^d$.

We considered both the Vietoris topology τ_V and the Fell topology τ_F.

Vietoris	Fell
+ Familiar for topologists	+ τ_F is Polish and compact
+ It is "the same" topology we put on $\mathbf{K}([0,1]^d)$	+ Generates a standard Borel space
- Not metrizable if the ambient space is not compact	± Coarser than Vietoris topology

Stability of the Fourier dimension

Fourier dimension is (in general) not stable under countable unions and not inner-regular for compacts ((Ekström, Persson, and Schmeling 2015)).

$$\dim_{\mathrm{F}}\left(\bigcup_n A_n\right) \neq \sup_n \dim_{\mathrm{F}}(A_n)$$

$$\dim_{\mathrm{F}}(A) \neq \sup\{\dim_{\mathrm{F}}(K) \,:\, K \subset A \text{ and } K \text{ is compact}\}$$

There is $G = \bigcup_n K_n$ with $\dim_{\mathrm{F}}(G) = 1$ and $\dim_{\mathrm{F}}(K_n) = 0$.

Theorem (Marcone, V.)

For every pointclass $\boldsymbol{\Gamma}$ *and every non-empty* $A \in \boldsymbol{\Gamma}(\mathbb{R}^d)$,

$$\dim_{\mathrm{F}}(A) = \sup\{\dim_{\mathrm{F}}(K) \,:\, K \subsetneq A \textit{ is bounded and } K \in \boldsymbol{\Gamma}(\mathbb{R}^d)\}.$$

Salem sets in $\mathbb{R}^d$

Theorem (Marcone, V.)

Fix $d \geq 1$. For every $p < d$ the sets

$$\{A \in \mathbf{F}(\mathbb{R}^d) \,:\, \dim_{\mathcal{H}}(A) > p\},$$
$$\{A \in \mathbf{F}(\mathbb{R}^d) \,:\, \dim_{\mathrm{F}}(A) > p\}$$

are $\mathbf{\Sigma}^0_2$-complete. Moreover, for every $q > 0$ the sets

$$\{A \in \mathbf{F}(\mathbb{R}^d) \,:\, \dim_{\mathcal{H}}(A) \geq q\},$$
$$\{A \in \mathbf{F}(\mathbb{R}^d) \,:\, \dim_{\mathrm{F}}(A) \geq q\},$$
$$\{A \in \mathbf{F}(\mathbb{R}^d) \,:\, A \in \mathscr{S}(\mathbb{R}^d)\}$$

are $\mathbf{\Pi}^0_3$-complete.

Effectivizations

All the results obtained are actually effective:

Theorem (Marcone, V.)

Let X be $[0,1]^d$ or $\mathbb{R}^d$ for some $d \geq 1$.

- $\{(A,p) \in \mathbf{F}(X) \times [0,d) : \dim_{\mathcal{H}}(A) > p\}$ *is* Σ^0_2*-complete*
- $\{(A,p) \in \mathbf{F}(X) \times (0,d] : \dim_{\mathcal{H}}(A) \geq p\}$ *is* Π^0_3*-complete*
- $\{(A,p) \in \mathbf{F}(X) \times [0,d) : \dim_{\mathrm{F}}(A) > p\}$ *is* Σ^0_2*-complete*
- $\{(A,p) \in \mathbf{F}(X) \times (0,d] : \dim_{\mathrm{F}}(A) \geq p\}$ *is* Π^0_3*-complete*
- $\{A \in \mathbf{F}(X) : A \in \mathscr{S}(X)\}$ *is* Π^0_3*-complete*

Effective measurability

f is called Σ^0_k**-measurable** iff preimages $f^{-1}(U)$ of open sets are Σ^0_k (relatively to $\mathrm{dom}(f)$).

f is called **effectively** Σ^0_k**-measurable** if the preimage can be uniformly computed from a name of U.

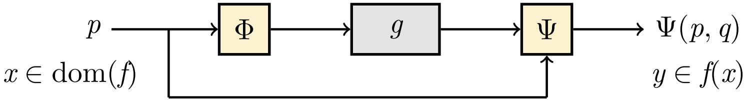
Our results imply that the maps $\dim_{\mathcal{H}}$ and $\dim_{\mathrm{F}}$ are effectively Σ^0_3-measurable.

Weihrauch reducibility

A **represented space** is a pair (X, δ_X) where X is just a set and $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ is a surjection.

If f and g are (possibly partial) multivalued functions, we say that f is **Weihrauch reducible** to g ($f \le_{\mathrm{W}} g$) if there are computable $\Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ s.t.

- If p is a name for $x \in \mathrm{dom}(f)$ then $\Phi(p)$ is a name for $z \in \mathrm{dom}(g)$
- If q is a name for $w \in g(z)$ then $\Psi(p, q)$ is a name for $y \in f(x)$



Operations on problems

We can define operations on multivalued functions to combine problems $(\times, {}^*, \hat{}, \ldots)$

In particular $f * g$ (intuitively) captures the idea of applying g, then do some computable operation (using the output of g) and then apply f.

We write $f^{[n]}$ for $\underbrace{f * f * ...f}_{n \text{ times}}$.

Weihrauch degree of $\dim_F$

Let lim be the problem of finding the limit in the Baire space.

Theorem ((Brattka 2005))

f is effectively Σ^0_{k+1}-measurable iff f is Weihrauch reducible to $\mathsf{lim}^{[k]}$

This is intuitively saying that $\mathsf{lim}^{[k]}$ is “the most complicated” effectively Σ^0_{k+1}-measurable problem.

We can think of $\mathsf{lim}^{[k]}$ as the problem of answering countably many Σ^0_k questions in parallel.

Weihrauch degree of $\dim_{\mathrm{F}}$

Theorem (Marcone, V.)

$\dim_{\mathrm{F}} \equiv_{\mathrm{W}} \mathsf{lim}^{[2]}$

Proof (Sketch).

$\leq_{\mathrm{W}}$: follows from (Brattka 2005), as our results imply that $\dim_{\mathrm{F}}$ is effectively Σ^0_3-measurable.

$\geq_{\mathrm{W}}$: given a sequence $(x_i)_{i\in\mathbb{N}}$ in $2^{\mathbb{N}}$, we can uniformly build a closed Salem subset A of $[0,1]^d$ s.t.

$$\dim(A) = \sum_{i\in\mathbb{N}} 2^{-i}\chi_{Q_2}(x_i)$$

In particular, it uniformly computes whether $x_i \in Q_2$. $\square$

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