Current thoughts on Hindman's Theorem

Jeffry Hirst
Appalachian State University
Boone, NC

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CTA Seminar

Hindman's Theorem

Hindman's theorem is a coloring result proved by Neil Hindman in his 1974 paper [2].

HT: If $f : \mathbb{N} \to k$ is a finite coloring of $\mathbb{N}$, then there is an infinite set whose finite sums are monochromatic. That is, there is an infinite set H and a color c such that for every finite set $\{h_0, h_1 \dots h_n\} \subset H$, we have $f(h_0 + h_1 + \cdots + h_n) = c$.

Hindman's theorem can be iterated for sequences of colorings.

IHT: If $\langle f_i \rangle_{i \in \mathbb{N}}$ is a sequence of finite colorings of $\mathbb{N}$, then there is a set $H = \{h_0 < h_1 < h_2 \dots\}$ such that for each j, the finite sums of $\{h_n \mid n \geqslant j\}$ are monochromatic for f_j.

Reverse Math bounds

Blass, Hirst, and Simpson [1] gave bounds for the strength of HT and IHT:

IHT is provable in ACA_0^+ (which is arithmetical comprehension plus an axiom asserting that every set has an ω jump.)

Over RCA_0, HT implies ACA_0.

The exact strength of HT and IHT remain open.

Related work:

Towsner [6] describes computable colorings with complicated monochromatic sets.

Montalban and Shore [4] prove conservation results over IHT.

Colorings and countable Boolean algebras

Given any finite partition (coloring) of $\mathbb{N}$, there is a countable Boolean *shift algebra* that contains:

- the partition (monochromatic) sets,
- all the finite sets,
- unions, intersections, and complements of sets in the algebra, and
- the shift sets $X - n = \{x - n \mid x \in X \wedge x \geqslant n\}$.

ACA_0 suffices to construct a nice presentation of the shift algebra for a coloring, with the sets given as a nonrepeating sequence of characteristic functions with maps from the indices of component sets to indices of unions, intersections, complements, and shifts. The algebras in [3] are not so nice.

Ultrafilters

A subset u of (the indices of the sets in) a shift algebra is called an *ultrafilter* if:

- $\emptyset$ is not in u,
- u is closed under intersection,
- if $X \supset Y \in u$, then $X \in u$, and
- for all X in the shift algebra, either $X \in u$ or $X^c \in u$.

The characteristic function for an ultrafilter u can be viewed as an element of $2^{\mathbb{N}}$.

The collection of all ultrafilters on a countable shift algebra form a closed subset of $2^{\mathbb{N}}$.

The collection of all ultrafilters containing a particular element of the shift algebra form a basic open subset of the closed set of ultrafilters.

Invariant ultrafilters

An ultrafilter u on a shift algebra is (almost downward translation) *invariant* if for every $X \in u$ there is an $n \in X$ such that $X - n \in u$.

When the shift algebra contains all the finite sets, u is invariant iff for every $X \in u$ and every $j \in \mathbb{N}$, there is an $n > j$ such that $n \in u$ and $X - n \in u$.

If the underlying shift algebra is the entire power set of $\mathbb{N}$, u is invariant iff

> for every $X \in u$, $\{n \mid X - n \in u\} \in u$, which happens iff u is an idempotent for Galvin-Glazer addition.

Invariant ultrafilters and monochromatic sets

Given a shift algebra constructed from a coloring, we can extract a monochromatic set closed under finite sums from any invariant ultrafilter.

Start with a monochromatic set in the ultrafilter, and shift and intersect in the following fashion.

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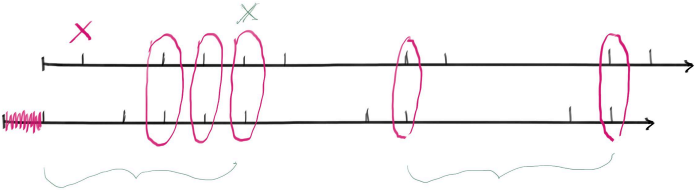

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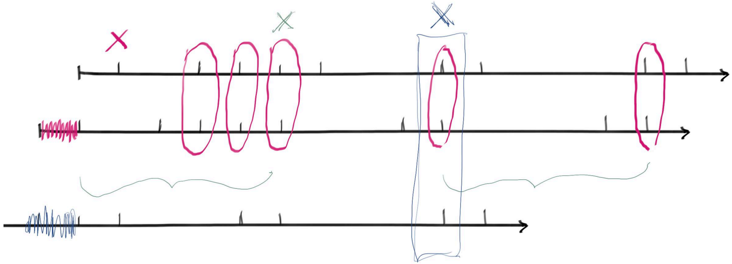

Invariant ultrafilters and IHT

Using an invariant ultrafilter for a shift algebra constructed from a sequence of colorings, we can dovetail to find a monochromatic set for the iterated Hindman's theorem.

Theorem [3] RCA_0 proves the following are equivalent:

1. IHT.
2. Every countable shift algebra has an invariant ultrafilter.

Summarizing: If we can find invariant ultrafilters in ACA_0, then we have proven IHT in ACA_0.

Shifting ultrafilters

Shifts and set operations interact nicely. For example,

$$(X - n) \cap (Y - n) = (X \cap Y) - n \text{ and}$$

$$(X - n)^c = X^c - n.$$

Suppose u is an ultrafilter. Define $u + 1$ by

$$X \in u + 1 \text{ if and only if } X - 1 \in u.$$

Then $u + 1$ is also an ultrafilter.

If u and v agree on many sets, then so do $u + 1$ and $v + 1$. In fact, the function sending each u to $u + 1$ is a continuous function on the space of ultrafilters.

Topological dynamics

Here is a lemma from Blass, Hirst, and Simpson [1]:

Lemma 5.10: The following is provable in ACA_0. Let U be a compact metric space and let $S : U \to U$ be a continuous function. For all $x \in U$ there exists a u in the orbit closure $\bar{x}$ such that every $v \in \bar{u}$ is uniformly recurrent.

What does this mean in our setting?

The function S defined by $S(u) = u + 1$ is continuous. Lemma 5.10 says there is an ultrafilter u such that every v in the closure of $\{u, S(u), S(S(u)), S^{(3)}(u), \ldots\}$ is uniformly recurrent.

v is uniformly recurrent means that for every k there is an m such that for every n there is a $j \leqslant m$ such that $S^{n+j}(v)$ and v agree on the first k sets. Informally, v is revisited remarkably regularly.

Orbits and invariant ultrafilters

Suppose that we have an ultrafilter u as provided by Lemma 5.10, whose orbit closure consists of uniformly recurrent ultrafilters.

Is there an invariant ultrafilter in the orbit closure?

Sometimes, yes. For example, if the orbit is finite, then there must be an invariant ultrafilter. For some infinite orbits, we can carry out another construction, based on witness sets.

Witness sets

We say that a set $W = \{w_0, w_1, w_2, \ldots\}$ is a witness set if there is an ultrafilter v such that W is in v and for every n, the shift set $W - w_n$ is not in v. Thus W witnesses that v is not invariant.

Associated with a witness set, there is a nested sequence of clopen sets:

All ultrafilters containing W

All ultrafilters containing $W \cap (W - w_0)^c$

All ultrafilters containing $W \cap (W - w_0)^c \cap (W - w_1)^c$

and so on.

The intersection of these sets is the closed set consisting of all non-invariant ultrafilters witnessed by W.

Witness sets and orbit closures

Our definition of witness set looks like a Σ^1_1 formula:

A set $W = \{w_0, w_1, w_2, \dots\}$ is a witness set if there is an ultrafilter v such that W is in v and for every n, the shift set $W - w_n$ is not in v.

However, in the orbit closure of u, for every n there is a j such that

$$W \cap (W - w_0)^c \cap (W - w_1)^c \cap \dots \cap (W - w_n)^c \in S^{(j)}(u)$$

if and only if the orbit closure of u contains a non-invariant ultrafilter v witnessed by W.

The set of witness sets is arithmetically definable.

If our boolean algebra is $\{B_0, B_1, B_2, \dots\}$ we say that B_j is a minimal witness set if B_j witnesses that some ultrafilter v is invariant and no B_i for $i < j$ witnesses that v is invariant. The set of minimal witnesses is also arithmetically definable.

Minimal witnesses and a sequence of open sets

Suppose $W_0, W_1 \ldots$ is the sequence of minimal witnesses for the orbit closure of u.

Sequentially select open sets associated with witnesses so small that they omit forward orbit elements with witnesses of arbitrarily large index.

If this process never halts, then we have a sequence of open sets such that:

- no initial segment is a cover of the orbit closure of u, and
- every non-invariant ultrafilter is contained in some open set in the sequence.

The orbit closure is a closed set, so by the Heine-Borel theorem, some ultrafilter lies outside the sequence of open sets. Thus some ultrafilter is invariant.

What is left?

What if the process described above halts?

In particular, what if there are only finitely many minimal witnesses?

Is there a way to use uniform recurrence to show that when the process halts there is an invariant ultrafilter? (In ACA_0?)

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