# On normal numbers

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Computability and Mathematical Definability
Celebrating the Seventieth Birthday of Theodore Slaman
October 11-13, 2024, University of California, Berkeley

1. V. Becher, P.A. Heiber and T. Slaman.
   A polynomial-time algorithm for computing absolutely normal numbers,
   *Information and Computation* 232: 1–9, 2013.

2. V. Becher, P.A. Heiber and T. Slaman.
   Normal numbers and the Borel hierarchy,
   *Fundamenta Mathematicae* 226: 63-77, 2014.

3. V. Becher and T. Slaman.
   On the normality of numbers to different bases,
   *Journal of the London Mathematical Society* 90 (2): 472–494, 2014.

4. V. Becher, P.A. Heiber and T. Slaman.
   A computable absolutely normal Liouville number,
   *Mathematics of Computation* 84(296): 2939–2952, 2015.

5. V. Becher, Y. Bugeaud and T. Slaman.
   On simply normal numbers to different bases,
   *Mathematische Annalen*, 364(1): 125-150, 2016.

6. C. Aistleitner, V. Becher and A.-M. Scheerer and T. Slaman.
   On the construction of absolutely normal numbers,
   *Acta Arithmetica* 180(4): 333–346, 2017.

7. V. Becher, Y. Bugeaud and T. Slaman.
   The irrationality exponents of computable numbers,
   *Proceedings of American Mathematical Society* 144:1509–1521, 2016.

8. V. Becher, J. Reimann and T. Slaman.
   Irrationality Exponent, Hausdorff Dimension and Effectivization,
   *Monatshefte fr Mathematik*, 185(2):167–188, 2018.

# Normal numbers

A base is an integer greater than or equal to $2$.

For a real number $x$, the expansion of $x$ in base $b$ is a sequence $a_1a_2a_3\ldots$ of integers from $\{0, 1, \ldots, b-1\}$ such that

$$x = \lfloor x \rfloor + \sum_{k\geq 1} a_k b^{-k} = \lfloor x \rfloor + 0.a_1a_2a_3\ldots$$

where infinitely many of the $a_k$ are not equal to $b-1$.

# Normal numbers

Definition (Borel, 1909)

A real number $x$ is simply normal to base $b$ if, in the expansion of $x$ in base $b$, each digit occurs with limiting frequency equal to $1/b$.

A real number $x$ is normal to base $b$ if $x$ is simply normal to base $b^k$, for every positive integer $k$.

A real number $x$ is absolutely normal if $x$ is normal to every base.

Theorem (Borel 1922, Niven and Zuckerman 1951)

*A real number $x$ is normal to base $b$ if, for every $k \geq 1$, every block of $k$ digits occurs in the expansion of $x$ in base $b$ with limiting frequency $1/b^k$.*

Two integers are multiplicatively dependent if one is a rational power of the other. Example: $2$ and $8$ are dependent but $2$ and $6$ are independent.

Theorem (Maxfield 1953)

*Let $b$ and $b'$ multiplicatively dependent. For any real number $x$, $x$ is normal to base $b$ if and only if $x$ is normal to base $b'$.*

# Not normal

$0.01\,002\,0003\,00004\,000005\,0000006\,00000007\,000000008\ldots$
is not simply normal to base $10$.

$0.0123456789\;\;0123456789\;\;0123456789\;\;0123456789\;\;0123456789\ldots$
is simply normal to base $10$, but not simply normal to base $100$.

The numbers is the middle third Cantor set are not simply normal to base $3$.

The rational numbers are not normal to any base.

# Existence of normal numbers

Theorem (Borel 1909)

*The set of absolutely normal numbers in the unit interval has Lebesgue measure* $1$.

**Borel 1909:**

**Give an example of (absolutely) normal number**

# Absolutely normal, non-effective constructions

Bulletin de la Société Mathématique de France (1917) 45:127–132; 132–144

**DÉMONSTRATION ÉLÉMENTAIRE DU THÉORÈME DE M. BOREL SUR LES NOMBRES ABSOLUMENT NORMAUX ET DÉTERMINATION EFFECTIVE D'UN TEL NOMBRE;**

PAR M. W. SIERPINSKI.

On appelle, d'après M. Borel, *simplement normal* par rapport à la base $q$ (1) tout nombre réel $x$ dont la partie fractionnaire

(1) E. BOREL, *Leçons sur la théorie des fonctions*, p. 197, Paris, 1914.

**SUR CERTAINES DÉMONSTRATIONS D'EXISTENCE;**

PAR M. H. LEBESGUE.

Dans une lettre, adressée à M. Borel, et qui accompagnait l'envoi de l'article précédent, M. Sierpinski se demandait si cet article devait être publié, s'il ne ferait pas double emploi avec une démonstration que j'avais indiquée à M. Borel et que celui-ci a signalée dans la deuxième édition de ses *Leçons sur la théorie des fonctions* (p. 198).

V. Becher and S. Figueira. An example of a computable absolutely normal number, 2002.

# Normal to a given base

Theorem (Champernowne, 1933)

$0.123456789101112131415161718192021\ldots$ *is normal to base* $10$.

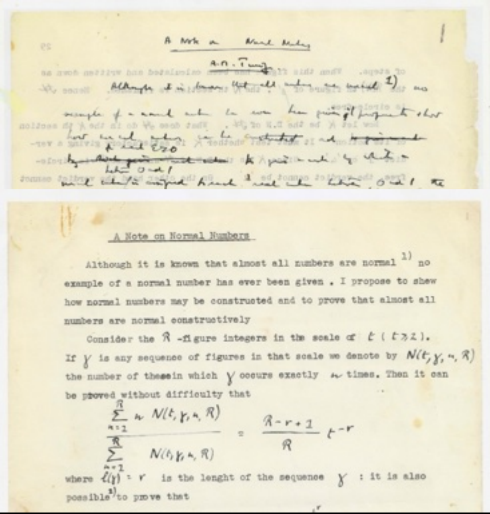
The proof is by direct counting.
It is unknown if it is normal to bases that are not powers of $10$.

Generalizations:

squares Besicovitch 1935,
primes Copeland and Ërdos 1946,
de Bruijn words Ugalde, 2000.

# Absolutely normal, effective-construction

Alan Turing, A note on normal numbers. Collected Works, Pure Mathematics, J.L. Britton editor, 1992.



A Note on Normal Numbers

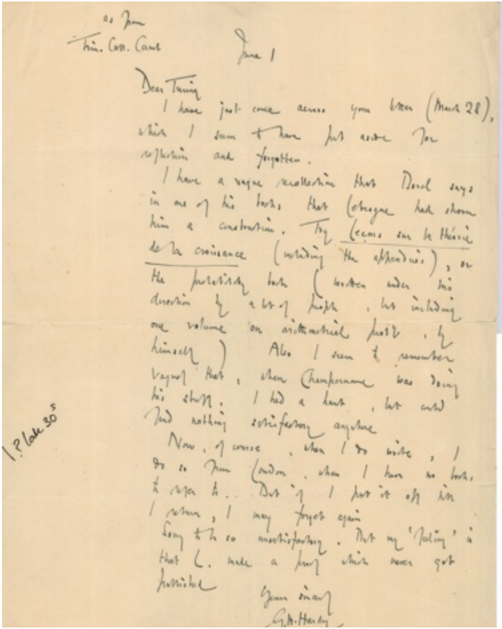
Although it is known that almost all numbers are normal [1] no example of a normal number has ever been given. I propose to shew how normal numbers may be constructed and to prove that almost all numbers are normal constructively

Consider the $R$-figure integers in the scale of $t$ ($t \geq 2$). If $\gamma$ is any sequence of figures in that scale we denote by $N(t,\gamma,n,R)$ the number of these in which $\gamma$ occurs exactly $n$ times. Then it can be proved without difficulty that

$$\frac{\sum_{n=1}^{R} n\, N(t,\gamma,n,R)}{\sum_{n=1}^{R} N(t,\gamma,n,R)} = \frac{R-r+1}{R}\, t^{-r}$$

where $l(\gamma) = r$ is the lenght of the sequence $\gamma$ : it is also possible [2] to prove that

Corrected and completed V. Becher and S.Figueira and R. Picchi. Turing's unpublished algorithm for normal numbers, 2007.


Trin. Coll. Camb

June 1

Dear Turing

I have just come across your letter (March 28), which I seem to have put aside for reflection and forgotten.

I have a vague recollection that Borel says in one of his books, that Lebesgue had shown him a construction. Try Leçons sur la théorie de la croissance (including the appendices), or the probability book (written under his direction by a lot of people, but including one volume on arithmetical prob, by himself) Also I seem to remember vaguely that, when Champernowne was doing his stuff, I had a hunt, but could find nothing satisfactory anywhere

Now, of course, when I do write, I do so from London, where I have no books, to refer to. But if I put it off till I return, I may forget again

Sorry to be so unsatisfactory. But my 'feeling' is that L. made a proof which never got published

Yours sincerely

G.H. Hardy

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G.H. Hardy

# Turing's algorithm for computing normal numbers

Theorem (Turing 1937?)

*An effective version of Borel's theorem for the Lebesgue measure of the set of absolutely normal numbers.*

Turing gives the following construction. For each $k, n$,

- $E_{k,n}$ is a finite union of open intervals with rational endpoints.
- Measure of $E_{k,n}$ is equal to $1 - \frac{1}{k} + \frac{1}{k+n}$.
- $E_{k,n+1} \subset E_{k,n}$.

For each $k$, the set $\bigcap_n E_{k,n}$ has Lebesgue measure exactly $1 - \frac{1}{k}$ and consists entirely of absolutely normal numbers.

# Turing's algorithm for computing normal numbers

Theorem (Turing 1937?)

*There is an algorithm that, given an integer $k$ and an infinite sequence $\nu$ of zeros and ones, produces an absolutely normal number $\alpha(k,\nu)$ in the unit interval, expressed in base two.*

At each step, divide the current interval in two halves,
Choose the half that includes normal numbers in large-enough measure.
The output $\alpha(k,\nu)$ is the trace of the left/right selection at each step.

Computation of the $n$-th digit requires doubly exponential in $n$ elementary operations.

Schmidt 1961/1962, Levin 1971 (proved in Alvarez and Becher 2015), gave other algorithms with exponential complexity.

**Turing 1937:**

**How to compute an absolutely normal number in polynomial time ?**

**($n$-th bit requiring polynomial in $n$ operations)**

# Algorithm in polynomial time

Ted Slaman's idea: just follow the measure.

Theorem (Becher, Heiber and Slaman, 2013)

*There is an algorithm that computes an absolutely normal number with just above quadratic time-complexity.*

The algorithm is based on Turing's.
Speed is gained by :

- testing the extension instead of the whole initial segment.
- slowing convergence to normality.

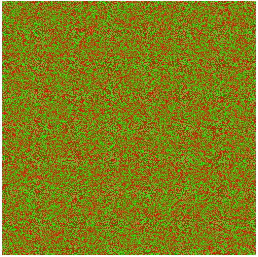
Lutz and Mayordomo (2013) and Figueira and Nies (2013) algorithm based on martingales.

Lutz and Mayordomo obtained in nearly linear time (2021)

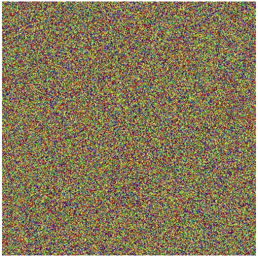
# Algorithm in polynomial time

Output of algorithm Becher, Heiber and Slaman, 2013 programmed by Martin Epszteyn.

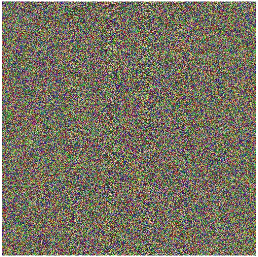
$$0.4031290542003809132371428380827059102765116777624189775110896366\ldots$$



base 2



base 6



base10

Plots of the first 250000 digits of the output of our algorithm.

Available from `http://www.dc.uba.ar/people/profesores/becher/software/ann.zip`

Computable absolutely Pisot normal numbers

MG Madritsch, AM Scheerer, RF Tichy 2016

Dynamical systems and uniform distribution of sequences

MG Madritsch, RF Tichy 2016

On absolutely normal and continued fraction normal numbers

V Becher, SA Yuhjtman 2019

Construction of $x$ such that $x$ and $1/x$ absolutely normal

V Becher, MG Madritsch 2022

Poisson generic sequences

N Álvarez, V Becher, M Mereb 2022

Poisson genericity in numeration systems with exponentially mixing probabilities

N Álvarez, V Becher, E Cesaratto, M Mereb, Y Peres, B Weiss 2024

**Alexander Kechris, before 1994:**

**What is the descriptive complexity of the set of (absoluteley) normal numbers?**

# The set of normal numbers

A real $x$ is normal to base $b$ if its expansion in base $b$, $0.x_1x_2x_3\ldots$, satisfies for every digit $d$ in $\{0,\ldots,b-1\}$,

$$\lim_{n\to\infty}\frac{\#\{j : x_j = d\}}{n} = \frac{1}{b},$$

Equivalently,

$$\forall d \in \{0,..,b-1\}\ \forall\varepsilon > 0 \exists m\ \forall n \geq m\ \left|\frac{\#\{j : x_j = d\}}{n} - \frac{1}{b}\right| \leq \varepsilon.$$

A real number $x$ is normal to base $b$ if

$$\forall\epsilon\exists m\forall n\ \varphi(x,b,n,\varepsilon)$$

where $\varphi$ has one free real variable $x$, one free integer variable $b$ and quantification only on integers.

The Borel hierarchy for subsets of the real numbers is the stratification of the $\sigma$-algebra generated by the open sets with the usual interval topology.

Theorem (Ki and Linton 1994)

*The set of real numbers that are normal to a fixed base is* $\mathbf{\Pi^0_3}$*-complete.*

Theorem (Becher, Heiber, Slaman 2014)

*The set of real numbers that are absolutely normal is* $\mathbf{\Pi^0_3}$*-complete.*

Achim Ditzen conjectured in 1994, we confirmed it:

**Theorem** (Becher and Slaman 2014)

*The set of real numbers normal to some base is $\boldsymbol{\Sigma^0_4}$-complete . The set of indices for computable numbers that are normal to at least one base is $\Sigma^0_4$-complete.*

**Theorem** (Becher and Slaman 2014)

*Fix a base s. There is a computable $f : \mathbb{N} \to \mathbb{Q}$ monotonically decreasing to $0$ such that for any $g : \mathbb{N} \to \mathbb{Q}$ monotonically decreasing to $0$, there is an absolutely normal number $x$ whose discrepancy for base $s$ eventually dominates $g$, and whose discrepancy for each base multiplicatively independent to $s$ is eventually dominated by $f$. Furthermore, $x$ is computable from $g$.*

**Theorem** (Becher and Slaman 2014, it answers Brown, Moran and Pearce 1985)

*For any given set of bases closed under multiplicative dependence, there are real numbers that are normal to each base in the given set, but not simply normal to any base in its complement.*

# A fixed point

The set of bases for which a real $x$ is normal can coincide with any arithmetical property on the set of integers (closed by multiplicative dependence) definable by a $\Pi^0_3$ formula relative to $x$ .

Theorem (Becher and Slaman 2014)

*For any $\Pi^0_3$ formula $\varphi$ in second order arithmetic there is a computable real number $x$ such that, for any non-perfect power $b$, $x$ is normal to base $b$ if and only if $\varphi(x, b)$ is true.*

# Tools

Theorem (Weyl's criterion)

*A sequence* $(x_n)_{n\geq 1}$ *of real numbers is uniformly distributed modulo one (u.d. mod 1) for Lebesgue measure if, and only if, for every non-zero integer* $t$, $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} e^{2\pi i t x_k} = 0.$

Theorem (Wall 1949)

*A number* $x$ *normal to base* $b$ *if and only if* $(b^k x)_{k\geq 0}$ *is u.d. mod 1 for Lebesgue measure. That is, if and only if, for every non-zero integer* $t$,
$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} e^{2\pi i t b^k x} = 0.$

# Tools

Normality to some bases, but not all of them, is also an almost-everywhere property, albeit not in the sense of Lebesgue.

Consider the Cantor set $C_s$ obtained by omitting the last digit (or two) in the base $s$ expansion of real numbers, with $s > 2$.

Clearly, no element of $C_s$ is simply normal to base $s$.

However Schmidt 1961/1962 (generalizing Cassels 1959) proved that almost all elements in $C_s$ for the uniform measure in $C_s$ are normal to every base $r$ multiplicatively independent to $s$.

## Theorem (Schmidt 1961/1962)

*For any given set $S$ of bases closed under multiplicative dependence, there are real numbers normal to every base in $S$ and not normal to any base in its complement. Furthermore, there is a real $x$ computable from $S$.*

Pollington 1981 showed that for each set $S$ the set of such numbers has full Hausdorff dimension.

# Tools

Let $\mu$ be a measure on $\mathbb{R}$. The Fourier transform $\hat{\mu}$ of $\mu$ is

$$\hat{\mu}(t) = \int_{-\infty}^{\infty} e^{2\pi i t x} d\mu(x).$$

If $\mu$ is a measure on $\mathbb{R}$ such that $\hat{\mu}$ vanishes at infinity sufficiently quickly then almost every real number is normal.

## Lemma (direct application of Davenport, Erdős, LeVeque's Theorem 1963)

*Let $\mu$ be a measure, $I$ an interval and $b$ a base. If for every non-zero integer $t$,*

$$\sum_{n\geq 1} \frac{1}{n} \int_I \left| \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i t b^k x} \right|^2 d\mu(x) \ < \infty$$

*then for $\mu$-almost all $x$ in interval $I$ are normal to base $b$.*

Deterministic numbers (Rauzy) are $\Pi_3^0$ complete.

D Airey, S Jackson, B Mance 2022

Continued fraction, cantor series expansions, $\beta$-expansions, generic-point in subshits with specification,

D Airey, S Jackson, D Kwietniak, B Mance 2020; D Airey, S Jackson, B Mance 2022

The base-$2$ normal but not normal to base-$3$ is $D_2(\Pi_3^0)$ complete
Moreover, the set of numbers that are continued fraction normal but not normal to *any* base-$b$ is $D_2(\Pi_3^0)$-hard, confirming the existence of uncountably many such numbers,

K. Beros 2017; S Jackson, B Mance, J Vandehey 2021

Poisson generic numbers are $\Pi_3^0$ complete $b$-normal that are not Poisson generic in base $b$ is is $D_2(\Pi_3^0)$

V. Becher, S Jackson, D Kwietniak, B Mance 2023.

**Bugeaud 2013:**

**What are the conditions on a set of bases so that there is a real number simply normal exactly to the bases in such a set?**

Theorem (Becher, Bugeaud and Slaman 2015)

*Set $S$ be a set of integers such that for each $b$ that is not a perfect power,*

*if $b^k \in S$ for some $k$, then $b^\ell \in S$ for every $\ell$ that divides $k$,*

*if $b^k \in S$ for infinitely many $k$, then $b^\ell \in S$ for all $\ell \geq 1$.*

*Then, there is a real $x$ simply normal to exactly the bases in $S$, and $x$ is computable from the se set $S$.*

*Moreover, the set of real numbers that satisfy this condition has full Hausdorff dimension.*

# Tools

Observation

*If $k$ is a multiple of $\ell$, simple normality to $b^k$ implies simple normality to $b^\ell$.*

Theorem (Long 1957)

*Simple normality to infinitely many powers of $b$ implies normality to base $b$.*

# Tools

For each base $b \in S$ that is not a perfect power, either all the powers of $b$ are in $S$, or just finitely many. This second case is challenging.

For each $b^n \notin S$ we construct a Cantor set such that almost every element (with respect to its uniform measure) is simply normal to each of the finitely many bases $b^m$ in $S$ and not simply normal to $b^n$.

We work with base $b^\ell$, where $\ell$ is a large common multiple of $n$ and the exponents $m$ for $b^m \in S$.

Among the numbers less than $b^\ell$, we find one or two, depending on the parity of $b$, which are balanced when written in any of the bases $b^m$ and which are unbalanced when written in base $b^n$. A combinatorial argument in modular arithmetic proves that such numbers less than $b^\ell$ exist.

The wanted Cantor set is for $b^\ell$ omitting these one or two digits.

- Normal numbers with digit dependencies

  C Aistleitner, V Becher, O Carton 2019

- Normal sequences with given limits of multiple ergodic averages

  L Liao, M Rams 2021

- On simply normal numbers with digit dependencies

  V Becher, A Marchionna, G Tenenbaum 2023

- Real numbers compressible in every base

  Nandakumar, Pulari 2023

**Borel 1950:**

**Conjecture**

**All irrational algebraic numbers are absolutely normal.**

**Bugeaud 2013:**

**Can you construct absolutely normal numbers with prescribed irrationality exponent?**

The irrationality exponent of a real number $x$, is the supremum of the set of real numbers $z$ for which the inequality $0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^z}$ is satisfied by an infinite number of integer pairs $(p, q)$ with positive $q$.

- Liouville numbers are the numbers with infinite irrationality exponent. Example: Liouville's constant $\sum_{n \geq 1} 10^{-n!}$.
- Almost all real numbers have irrationality exponent equal to $2$.
- Rational numbers have irrationality exponent equal to $1$.
- Irrational algebraic numbers have irrationality exponent equal to $2$. (Thue - Siegel - Roth theorem 1955).
- Every real greater than or equal to $2$ is the irrationality exponent of some real ( Jarnik 1931)

Theorem (Becher, Bugeaud and Slaman 2016)

*A real number $a \geq 2$ is the irrationality exponent of some computable real number if and only if $a$ is the upper limit of a computable sequence of rational numbers.*

Theorem (Becher, Reimann and Slaman 2018)

*Let $a$ be a real number greater than or equal to $2$. For every real number $b \in [0, 2/a]$ there is a Cantor-like set $E$ with Hausdorff dimension equal to $b$ such that, for the uniform measure on $E$, almost all real numbers have irrationality exponent equal to $a$.*

Theorem (Becher, Reimann and Slaman 2018)

*Let $a$ and $b$ be real numbers such that $a \geq 2$ and $b \in [0, 2/a]$. There is a Cantor-like set $E$ such that, for the uniform measure on $E$, almost all real numbers in $E$ have irrationality exponent equal to $a$ and effective Hausdorff dimension equal to $b$.*

Theorem (Bugeaud 2002)

*There is an absolutely normal Liouville number.*

Theorem (Becher, Heiber and Slaman 2015)

*There is a computable absolutely normal Liouville number.*

Theorem (unpublshed Becher, Bugeaud and Slaman 2015)

*For every real $a \geq 2$, there is an absolutely normal number computable in $a$ and with irrationality exponent equal to $a$.*

# Tools

If we consider appropriate measures, most elements of well structured sets are absolutely normal, unless the sets have evident obstacles.

- Jarník (1929) and Besicovich (1934) defined a Cantor-like set for reals with a given irrationality exponent.
- Kaufman (1981) defined a measure on Jarník's set whose Fourier transform decays quickly.
- Bluhm (2000) refined it into a measure supported by the Liouville numbers, whose Fourier transform decays quickly.

For the Liouville case, we tailored Bluhm's measure for effective approximations. Support consists entirely of absolutely normal numbers.

For the case of finite irrationality exponent, we considered the uniform measure on the fractal set given by the central halves of Jarník's intervals. Support is strictly included in support of Kaufman's measure and consists entirely of absolutely normal numbers.

**Korobov 1955:**

**What is maximum decay of discrepancy of the first $N$ terms of $(b^n x)_{n\geq 0}$ in the theory of uniform distribution modulo one?**

For $(x_n)_{n\geq 1} \subset [0,1)^{\mathbb{N}}$, the discrepancy of the first $N$ terms

$$D_N((x_n)_{n\geq 1}) = \sup_{(a,b)\subset(0,1)} \left| \frac{\#\{n : 1 \leq n \leq N, x_n \in (a,b)\}}{N} - (b-a) \right|.$$

Thus, $(x_n)_{n\geq 1}$ us u.d. exactly when $\lim_{N\to\infty} D_N((x_n)_{n\geq 1}) = 0$.

## Theorem (Schmidt 1972)

*There is constant $C$ such that for every $(x_n)_{n\geq 1} \subset [0,1)^{\mathbb{N}}$ such that there are infinitely many $N$s with $D_N((x_n)_{n\geq 1}) > C(\log N)/N$.*

This is sharp.

# Normality as discrepancy going to $0$

A real number $x$ is normal to an integer base $b \geq 2$ if the sequence $(b^n x)_{n\geq 1}$ is uniformly distributed modulo 1; that is, if $D_N((b^n x \mod 1)_{n\geq 1}$ goes to $0$ when $N$ goes to infinity.

Theorem (Gál , Gál 1964)

*There is $C$ for almost all real numbers $x$,*

$$\limsup_{N\to\infty} \frac{\sqrt{N}}{\log logN} D_N((b^n x \mod 1)_{n\geq 1}) < C.$$

Question (Korobov 1955)

*Let integer $b \geq 2$. Find a function $f : \mathbb{N} \to \mathbb{R}$ with maximum decay such that there is a real number $x$ for which*

$$D_N((b^n x \mod 1)_{n\geq 0}) < f(N), \text{ for } N = 1, 2, 3, ....$$

## Theorem (Levin 1979)

*The is a constructions of $x \in \mathbb{R}$ such that for every $b \geq 2$,*
$D_N((b^n x \mod 1)_{n\geq 0}$ *is* $O((\log N)^3 N^{1/2})$.

## Theorem (Aistleitner, Becher, Scheerer and Slaman 2017)

*We give a construction of $x \in \mathbb{R}$ such that $b \geq 2$,*
$D_N((b^n x \mod 1)_{n\geq 0}$ *is* $O(N^{1/2})$.

## Theorem (Levin 1999, Hofer and Larcher 2022,2023)

*There is a construction of $x \in \mathbb{R}$ such that*
$D_N((2^n x \mod 1)_{n\geq 0})$ *is* $O((\log N)^2))/N$
*and there is $C$ and infinitely many $N$s, $D_N > C(\log N)^2/N$.*

Becher and Carton 2019 generalized Levin's constructioin terms of nested perfect necklaces.

**From Korobov's question 1955 and known results:**

**Is there a real number $x$ such that $D_N((2^n x \mod 1)_{n\geq 0})$ is $o((\log N)^2)/N$?**

**Happy birthday Ted!**