On a question of Slaman and Steel

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Is there a degree invariant solution to Post's problem?

Open Problem (Sacks, 1966)

Is there an $e \in \omega$ *so that for all* $x \in 2^\omega$,

$$x < W_e^x < x'$$

and for all $x, y \in 2^\omega$,

$$x \equiv_T y \rightarrow W_e^x \equiv_T W_e^y?$$

Martin's conjecture

Recall a **Turing cone** is a set of the form $\{y \in : y \geq_T x\}$ for some degree x. Assuming $\mathsf{ZF} + \mathsf{AD}$, if $A \subseteq \mathscr{D}_T$, then either A contains a Turing cone, or the complement of A contains a Turing cone.

$f\colon 2^\omega \to 2^\omega$ is **Turing invariant** if $x \equiv_T y \to f(x) \equiv_T f(y)$.

Conjecture (Martin, 1970s, ZF + AD + DC)

I. *If $f\colon 2^\omega \to 2^\omega$ is Turing invariant, then either $f(x) \geq_T x$ on a cone, or $[f(x)]_T$ is constant on a cone.*

II. *The relation "$\leq_T$ on a cone" prewellorders the Turing invariant functions which are increasing on a cone, and successor is given by the Turing jump.*

By II if $f(x) \geq_T x$ on a cone, then either $f(x) \equiv_T x$ on a cone or $f(x) \geq_T x'$ on a cone which gives a negative answer to Sacks's question.

Slaman-Steel's results on Martin's conjecture

Say $f\colon 2^\omega \to 2^\omega$ is **uniformly Turing invariant** if there is a function $u\colon \omega^2 \to \omega^2$ so that if $x \equiv_T y$ via the Turing reductions d and e, then $f(x) \equiv_T f(y)$ vi $u(d, e)$.

E.g. the Turing jump is uniformly Turing invariant.

Theorem (Steel 1982, Slaman-Steel 1988)

Martin's conjecture is true for uniformly Turing invariant functions.

Lachlan (1975) had showed there is no uniformly Turing invariant solution to Post's problem.

Motivation for Martin's conjecture

Martin's conjecture was motivated in part by the wellfoundedness of the Wadge hierarchy. Recent results make Martin's motivation seem especially prescient.

Say $f\colon 2^\omega \to 2^\omega$ is $(\equiv_T, \equiv_m)$-invariant if $x \equiv_T y \to f(x) \equiv_m f(y)$.

Theorem (Kihara-Montalbán, 2018, ZF + DC + AD)

The uniformly $(\equiv_T, \equiv_m)$-invariant functions on 2^ω under the relation "$\leq_m$ on a cone" are prewellordered and are in bijective correspondence with Wadge degrees.

An attempt to build a counterexample

Say an equivalence relation E on 2^ω is hyperfinite if there is an increasing sequence $F_0 \subseteq F_1 \subseteq \ldots$ of equivalence relations on 2^ω with finite classes so that $\bigcup_n F_n = E$.

Theorem (Slaman-Steel, ZF + AD + DC)

If $\equiv_T$ is hyperfinite, then there is a counterexample to Martin's conjecture.

Construct $f\colon 2^\omega \to 2^\omega$ by forcing. At step n, extend the approximation of f to ensure $f(x)$ will be Cohen generic, and also make coding commitments so that if $x \mathrel{E_n} y$, then $f(x) \equiv_T f(y)$.

Theorem (Slaman-Steel, 1988, ZF + AD + DC)

$\equiv_T$ is not hyperfinite.

Hyperfiniteness

Hyperfiniteness is now a fundamental notion in descriptive set theory.

Theorem (Slaman-Steel, 1988, $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}^+$)

An equivalence relation E is hyperfinite iff it is induced by an action of the group $\mathbb{Z}$ of integers.

This theorem is the first in an investigation of what other groups have this property. As well as many other applications in descriptive set theory.

Borel reducibility of equivalence relations

A research program of descriptive set theory in the past three decades has been to understand the relative complexity of equivalence relations under Borel reducibility. If E and F are equivalence relations on the spaces X and Y, then we define $E \leq_B F$ if there is a Borel function $f\colon X \to Y$ so that for all $x, y \in X$, $x \mathrel{E} y \leftrightarrow f(x) \mathrel{F} f(y)$.

E.g. the Turing jump is a Borel reduction from $\equiv_T$ to $\equiv_m$:

$$x \equiv_T y \leftrightarrow x' \equiv_m y'$$

Countable Borel equivalence relations

- (Harrington-Kechris-Louveau 1990) for all CBERs E, either $E \leq_B =_{2^\omega}$, or $E_0 \leq_B E$, where E_0 is the equivalence relation of equality mod finite on 2^ω.
- (Dougherty-Jackson-Kechris 1994) E is (Borel) hyperfinite iff $E \leq_B E_0$.
- (Dougherty-Jackson-Kechris 1994) There is a **universal** countable Borel equivalence relation E_∞, so for all countable Borel equivalence relations E, $E \leq_B E_\infty$.
- (Slaman-Steel) Arithmetic equivalence is universal.
- Conjecture (Kechris, 1999) $\equiv_T$ is a universal countable Borel equivalence relation. **This contradicts Martin's conjecture!** A Borel reduction from $\equiv_T \sqcup \equiv_T$ to $\equiv_T$ would give two Turing invariant functions with disjoint ranges. So at most one could contain a cone, so the other must be constant on a cone. Contradiction!

Slaman and Steel's question

By Slaman-Steel's theorem that $\equiv_T$ is not hyperfinite, if we write $\equiv_T$ as an increasing union of CBERs $\bigcup_n F_n$ where $F_0 \subseteq F_1 \subseteq \ldots$, then there is some x so that $[x]_{F_n}$ contains an infinite set $\{y_0, y_1, \ldots\}$.

A Sacksian question: how hard is it to define such an infinite sequence $(y_n)_{n\in\omega}$ from x?

Question (Slaman-Steel 1988, Is $\equiv_T$ hyper-recursively-finite?)

Can we write Turing equivalence $\equiv_T$ as an increasing union of CBERs $F_0 \subseteq F_1 \subseteq \ldots$ such that no equivalence class $[x]_{F_n}$ contains an infinite set $\{y_n : n \in \omega\}$ where $(y_n)_{n\in\omega}$ is uniformly computable from x?

The robustness of Slaman and Steel's question

More generally, suppose E is an equivalence relation on X, and $f_i\colon X \to X^\omega$ is a Borel function for every $i \in \omega$. Say E is (f_i)-**finite** if no E-class $[x]_E$ contains an infinite sequence of the form $f_i(x)$.

Say that E is **hyper-**(f_i)**-finite** if we can write $E = \bigcup_m F_m$ as an increasing union of equivalence relations that are (f_i)-finite.

Lemma (Day-M.)

The following are equivalent.

1. *Turing equivalence is hyper-recursively-finite*
2. *For every CBER E on 2^ω and every sequence $(f_i)_{i\in\omega}$ of Borel functions $f_i\colon 2^\omega \to 2^\omega$, E is hyper-(f_i)-finite.*

Proof idea: let $\alpha < \omega_1$, be sufficiently large. Pull back a witness to hyper-recursive-finiteness of $\equiv_T$ along the map $x \mapsto x^{(\alpha)}$.

Consequences of a positive answer

Theorem (Day-M.)

Suppose $\equiv_T$ is hyper-recursively-finite. Then is a $(\equiv_T, \equiv_m)$-invariant function $f : 2^\omega \to 2^\omega$ which is not uniformly invariant on any pointed perfect set, and such that $f(x) \not\geq_m x'$ and $x' \not\geq_m f(x)$.

That is, if Slaman and Steel's question has a positive answer, there is a counterexample to a version of Martin's conjecture for $(\equiv_T, \equiv_m)$-invariant functions, in the spirit of Kihara-Montalbán.

Theorem (Day-M.)

Suppose $\equiv_T$ is hyper-recursively-finite. Then there is a universal CBER that is not uniformly universal: $\equiv_m$ on 2^ω.

Uniformly universal CBERs were defined by Montalbán, Reimann, and Slaman.

Proof ideas:

First use the self-strengthening of Slaman-Steel's question to write $\equiv_T$ as an increasing union of equivalence relations containing no uniformly definable arithmetical sequence.

Then make a generic Turing invariant function $f\colon 2^\omega \to 2^\omega$ by forcing. At step n, extend the approximation of f to meet dense sets to diagonalize, and also make generic coding commitments so that if $x \mathrel{E_n} y$, then $f(x) \equiv_m f(y)$. The analysis of $f(x)$ boils down to analyzing finitely branching trees of attempts to iteratively decode how $f(x_n)$ is coded into $f(x_{n-1})$ is coded into $\ldots f(x)$.

The proof that $\equiv_m$ on 2^ω is a universal CBER uses a similar idea. (M. 2017) had already shown that $\equiv_1$ on 2^ω is not uniformly universal CBER.

A tool used in the proof

Two often used constructions in computability theory:

1. There is a Borel function $f\colon 2^\omega \to 2^\omega$ so that if $x_0, \ldots, x_n$ are distinct, then $f(x_0), \ldots, f(x_n)$ are mutually 1-generic.
2. There is a Borel function $f\colon 2^\omega \to 2^\omega$ so that for all x, $f(x)$ is x-generic.

It is impossible to have a Borel function f with both properties (1) and (2). If (2) holds, then $\mathrm{ran}(f)$ is nonmeager, which implies $\mathrm{ran}(f)$ contains two elements which are equal mod finite.

However, there is a Borel function such that

1. $x_0, \ldots, x_n$ distinct implies $f(x_0), \ldots, f(x_n)$ mutually 1-generic.

2'. For all $x \in 2^\omega$, $f(x)$ and x form a minimal pair.

Open: Does there exist a Borel function $f\colon 2^\omega \to 2^\omega$ so that for all distinct $x, y \in 2^\omega$ with $x \leq_T y$, $f(x)$ and $f(y)$ are mutually x-generic?

Slaman and Steel's question has a positive answer on generic or random reals

There is a comeager set on which Turing equivalence is hyperfinite, and hence hyper-recursively-finite. This is by the generic hyperfiniteness theorem of Hjorth-Kechris, Sullivan-Weiss-Wright, and Woodin.

Proposition (Day-M.)

If E is a countable Borel equivalence relation on X, μ is a Borel probability measure on X, and $\{f_i \colon X \to X^\omega : i \in \omega\}$ are Borel functions, then there is a μ-conull Borel set A so that $E \restriction A$ is hyper-(f_i)-finite.

A conjecture

We conjecture there is no way of nontrivially writing $\equiv_T$ as an increasing union.

Conjecture (Day-M.)

Suppose we write $\equiv_T$ as an increasing union of Borel equivalence relations $\bigcup_n E_n$. Then there is some n and some pointed perfect set $P \subseteq 2^{\mathbb{N}}$ so that $E_n \upharpoonright P = (\equiv_T \upharpoonright P)$.

Thanks for coming to my Ted talk!