Computable representations of exchangeable graphs

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Algebraic Closure and Definable Closure

Motivation

- **Algebraic closure** (acl) and **definable closure** (dcl) provide natural characterizations of a "neighborhood" of a set.
- A structure has **trivial dcl** when the dcl of every finite set is itself. (This is equivalent to having trivial acl.)
- The property of a structure having trivial dcl has played an important role in combinatorial model theory and descriptive set theory.

  Some characterizations in terms of this property:
  - universal graphs with forbidden subgraphs (Cherlin–Shelah–Shi, 1999)
  - invariant measures concentrated on an isomorphism class (AFP, 2016)
  - structurable equivalence relations (Chen–Kechris, 2018)

First Order Definable Closure

Suppose $\mathcal{A}$ is an $\mathcal{L}$-structure and $A_0 \subseteq \mathcal{A}$

The **first order definable closure** of A_0 is the smallest set $\mathrm{dcl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}(A_0) \subseteq \mathcal{A}$ containing A_0 such that whenever

- $\varphi(\overline{x}; \overline{y})$ is a first order $\mathcal{L}$-formula,
- $\overline{a} \in A_0$ is of the same type as $\overline{x}$,
- $|\{\overline{b} : \mathcal{A} \models \varphi(\overline{a}; \overline{b})\}| = 1$, and
- $\mathcal{A} \models \varphi(\overline{a}; \overline{b})$

then $\overline{b} \subseteq \mathrm{dcl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}(A_0)$.

Lemma

$\mathrm{dcl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})} = \mathrm{dcl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}(\mathrm{dcl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}(A_0))$.

In particular $\mathrm{dcl}_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}(A_0)$ is the smallest set containing A_0 closed under application of $\mathrm{dcl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}$.

Example of First Order Definable Closure

Example

If $\mathcal{L}$ is a language with functions and $\mathcal{M}$ is a $\mathcal{L}$-structure. Then for any $A_0 \subseteq M$ we have $\mathrm{dcl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}(A_0)$ contains the functional closure of A_0.

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Example

Consider the structure $(\mathbb{N}, S)$ where S is the binary relation which holds precisely on $S(a, a+1)$ for $a \in \mathbb{N}$.

We then have $\mathrm{dcl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}(\emptyset) = \mathbb{N}$.

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We then have $\mathrm{dcl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}(\emptyset) = \mathbb{N}$.

Example

Consider the structure $(\mathbb{Z}, S)$ where S is the binary relation which holds precisely on $S(a, a+1)$ for $a \in \mathbb{Z}$.

We then have $\mathrm{dcl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}(\emptyset) = \emptyset$ as there is an automorphism taking any a to b in $\mathbb{Z}$.

Further for any element $z \in \mathbb{Z}$, $\mathrm{dcl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}(\{z\}) = \emptyset$.

First Order Algebraic Closure

Suppose $\mathcal{A}$ is an $\mathcal{L}$-structure and $A_0 \subseteq \mathcal{A}$

The **first order algebraic closure** of A_0 is the smallest set $\mathrm{acl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}(A_0) \subseteq \mathcal{A}$ containing A_0 such that whenever

- $\varphi(\overline{x}; \overline{y})$ is a first order $\mathcal{L}$-formula,
- $\overline{a} \subseteq A_0$ is of the same type as $\overline{x}$,
- $\{\overline{b} : \mathcal{A} \models \varphi(\overline{a}; \overline{b})\}$ is finite, and
- $\mathcal{A} \models \varphi(\overline{a}; \overline{b})$

then $\overline{b} \subseteq \mathrm{acl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}(A_0)$.

Lemma

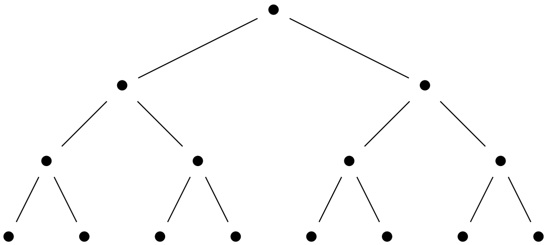
$\mathrm{acl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})} = \mathrm{acl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}(\mathrm{acl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}(A_0))$.

In particular $\mathrm{acl}_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}(A_0)$ is the smallest set containing A_0 closed under application of $\mathrm{acl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}$.

Example of First Order Algebraic Closure

Example

Consider the binary tree



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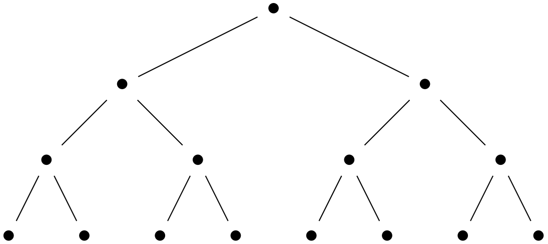

Example

Suppose K is a an algebraicly closed field. Then for any $K_0 \subseteq K$, $\mathrm{acl}^1_{\mathcal{L}_{\omega,\omega}(\mathcal{L})}(K_0)$ is the algebraic closure of K_0.

Background

Multi-Sorted Languages

We work with **many-sorted** languages and structures.

(We could instead encode each sort using a unary relation symbol. This would not affect most of our results, but we are also interested in how model-theoretically complicated the structures we build are, and if we do not allow sorts then our lower bound on the complexity of algebraic closure will not yield an $\aleph_0$-categorical structure.)

Let $\mathcal{L}$ be a (many-sorted) language, let $\mathcal{A}$ be an $\mathcal{L}$-structure, and suppose that $\bar{a}$ is a tuple of elements of $\mathcal{A}$. We say that the **type** of $\bar{a}$ is $\prod_{i \leq n} X_i$ when $\bar{a} \in \prod_{i \leq n} (X_i)^{\mathcal{A}}$, where each of $X_0, \ldots, X_{n-1}$ is a sort of $\mathcal{L}$.

The type of a tuple of variables is the product of the sorts of its constituent variables (in order). The type of a relation symbol is defined to be the type of the tuple of its free variables, and similarly for formulas.

Write $(\forall \bar{x} : X)$ and $(\exists \bar{x} : X)$ to quantify over a tuple of variables $\bar{x}$ of type X.

Computable Languages and Structures

Let $\mathcal{L} = \big((X_j)_{j \in J},\, (R_i)_{i \in I}\big)$ be a language, where $I, J \in \mathbb{N} \cup \{\mathbb{N}\}$ and $(X_j)_{j \in J}$ and $(R_i)_{i \in I}$ are collections of sorts and relation symbols, respectively.

Let $\mathrm{ty}_{\mathcal{L}} \colon I \to J^{<\omega}$ satisfy $\mathrm{ty}_{\mathcal{L}}(i) = (j_0, \ldots, j_{n-1})$ for all $i \in I$, where the type of R_i is $\prod_{k<n} X_{j_k}$.

$\mathcal{L}$ is a **computable language** when $\mathrm{ty}_{\mathcal{L}}$ is a computable function. For each computable language, we fix a computable encoding of all first-order formulas of the language.

$\mathcal{A}$ is a **computable $\mathcal{L}$-structure** when it is an $\mathcal{L}$-structure with computable underlying set such that the sets $\{(a, j) : a \in X_j^{\mathcal{A}}\}$ and $\{(\overline{b}, i) : \overline{b} \in R_i^{\mathcal{A}}\}$ are computable subsets of the appropriate domains.

Codes for Computable Languages and Structures

We say $c \in \mathbb{N}$ is a **code for a structure** if

- $\{c\}(0)$ is a code for a computable language, and
- $\{c\}(1)$ is a code for a computable structure in that language.

In this case, write

- $\mathcal{L}_c$ for the language that $\{c\}(0)$ codes,
- $\mathcal{M}_c$ for the structure that $\{c\}(1)$ codes, and
- T_c for the first-order theory of $\mathcal{M}_c$.

We let CompStr be the collection of $c \in \mathbb{N}$ that are codes for structures.

Formula by formula analysis

Sets encoding 1-step algebraic or definable closure

- CL $:= \big\{(c, \varphi(\overline{x}; \overline{y}), \overline{a}, k) : c \in \text{CompStr}, \varphi(\overline{x}; \overline{y})$ a first-order $\mathcal{L}_c$-formula, $\overline{a} \in \mathcal{M}_c$ of the same type as $\overline{x}$, and $k \in \mathbb{N} \cup \{\infty\}$ with $|\operatorname{cl}_{\varphi, \mathcal{M}_c}(\overline{a})| = k\big\}$.
- ACL $:= \big\{(c, \varphi(\overline{x}; \overline{y}), \overline{a}) : (\exists k \in \mathbb{N})\,(c, \varphi(\overline{x}; \overline{y}), \overline{a}, k) \in \text{CL}\big\}$.
- DCL $:= \big\{(c, \varphi(\overline{x}; \overline{y}), \overline{a}) : (c, \varphi(\overline{x}; \overline{y}), \overline{a}, 1) \in \text{CL}\big\}$.
- For $Y \in \{\text{CL}, \text{ACL}, \text{DCL}\}$ and $n \in \mathbb{N}$ let

  $$Y_n := \{t \in Y : \text{the second coordinate of } t \text{ is a Boolean combination of } \Sigma_n\text{-formulas}\}.$$

- For $Y \in \{\text{CL}, \text{ACL}, \text{DCL}\} \cup \{\text{CL}_n, \text{ACL}_n, \text{DCL}_n\}_{n \in \mathbb{N}}$ and $c \in \text{CompStr}$, let $Y^c := \{u : (c)^\wedge u \in Y\}$, i.e., select those elements of Y whose first coordinate is c, and then remove this first coordinate.

Sets encoding algebraic or definable closure

Let $c \in \mathrm{CompStr}$, Φ be a set of first-order $\mathcal{L}_c$-formulas and $X \subseteq \mathcal{M}_c$.

Define $\mathrm{acl}^n_{\Phi,c}(X)$ for $n \in \mathbb{N}$ by induction as follows.

- $\mathrm{acl}^0_{\Phi,c}(X) := X$,
- $\mathrm{acl}^1_{\Phi,c}(X) := X \cup \bigcup\{\overline{b} \subseteq \mathcal{M}_c \ : \ (\exists\varphi(\overline{x};\overline{y}) \in \Phi)(\exists\overline{a} \subseteq B)$ $\mathcal{M}_c \models \varphi(\overline{a};\overline{b}) \wedge (c, \varphi(\overline{x};\overline{y}), \overline{a}) \in \mathrm{ACL}\}$,
- $\mathrm{acl}^{n+1}_{\Phi,c}(X) := \mathrm{acl}^1_{\Phi,c}\big(\mathrm{acl}^n_{\Phi,c}(X)\big)$.

Let $\mathrm{acl}_{\Phi,c}(X) := \bigcup_{i\in\mathbb{N}} \mathrm{acl}^i_{\Phi,c}(X)$.

Define $\mathrm{dcl}^n_{\Phi,c}(X)$ for $n \in \mathbb{N}$ by induction as follows.

- $\mathrm{dcl}^0_{\Phi,c}(X) := X$,
- $\mathrm{dcl}^1_{\Phi,c}(X) := X \cup \bigcup\{\overline{b} \subseteq \mathcal{M}_c \ : \ (\exists\varphi(\overline{x};\overline{y}) \in \Phi)(\exists\overline{a} \subseteq X)$ $\mathcal{M}_c \models \varphi(\overline{a};\overline{b}) \wedge (c, \varphi(\overline{x};\overline{y}), \overline{a}) \in \mathrm{DCL}\}$,
- $\mathrm{dcl}^{n+1}_{\Phi,c}(X) := \mathrm{dcl}^1_{\Phi,c}\big(\mathrm{dcl}^n_{\Phi,c}(X)\big)$.

Let $\mathrm{dcl}_{\Phi,c}(X) := \bigcup_{i\in\mathbb{N}} \mathrm{dcl}^i_{\Phi,c}(X)$.

In order to study the computability-theoretic content of the algebraic and definable closure operators, we will consider the following encodings of their respective graphs.

Definition

Let $c \in \mathsf{CompStr}$ and let Φ be a set of first-order $\mathcal{L}_c$-formulas. Define

$$\mathsf{acl}_{\Phi,c} := \{(a, A) : a \in \mathsf{acl}_{\Phi,c}(A) \text{ and } A \text{ is a finite subset of } \mathcal{M}_c\}$$

$$\mathsf{dcl}_{\Phi,c} := \{(a, A) : a \in \mathsf{dcl}_{\Phi,c}(A) \text{ and } A \text{ is a finite subset of } \mathcal{M}_c\}$$

Complexity of Algebraic and Definable Closure

The complexity of CL, ACL, and DCL

CompStr is a Π^0_2 class.

Therefore the sets $\mathrm{CL}, \mathrm{ACL}, \mathrm{DCL}$ must be computability-theoretically complicated.

We therefore instead consider how complex $\mathrm{CL}^c, \mathrm{ACL}^c, \mathrm{DCL}^c$ can be, when $c \in \mathrm{CompStr}$.

Relationships Between CL^c, ACL^c and DCL^c

Lemma

Uniformly in $c \in \mathsf{CompStr}$ and $n \in \mathbb{N}$, the set

$$\{(\varphi(\overline{x};\overline{y}), \overline{a}, k) \in \mathsf{CL}^c_n \ : \ k \in \mathbb{N}, \ k \geq 1\}$$

is computably enumerable from DCL^c_n.

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Lemma

Uniformly in $c \in$ CompStr and $n \in \mathbb{N}$, the set

$$\left\{\left(\varphi(\overline{x};\overline{y}), \overline{a}, k\right) \in \mathrm{CL}^c_n \ : \ k = 0\right\}$$

is computably enumerable from DCL^c_n.

Relationships Between CL^c, ACL^c and DCL^c

Lemma

Uniformly in $c \in \mathrm{CompStr}$ and $n \in \mathbb{N}$, the set

$$\{(\varphi(\overline{x};\overline{y}), \overline{a}, k) \in \mathrm{CL}_n^c \,:\, k \in \mathbb{N},\ k \geq 1\}$$

is computably enumerable from DCL_n^c.

Lemma

Uniformly in $c \in \mathrm{CompStr}$ and $n \in \mathbb{N}$, the set

$$\left\{\left(\varphi(\overline{x};\overline{y}), \overline{a}, k\right) \in \mathrm{CL}_n^c \,:\, k = 0\right\}$$

is computably enumerable from DCL_n^c.

Lemma

Uniformly in $c \in \mathrm{CompStr}$ and $n \in \mathbb{N}$, there are computable reductions in both directions between $\mathrm{ACL}_n^c \coprod \mathrm{DCL}_n^c$ and CL_n^c.

Relationships Between ACL^c and $\mathrm{acl}_{\Phi,c}$

Proposition

Uniformly in the parameter $c \in \mathrm{CompStr}$ and an encoding of a computable set Φ of Σ_n first-order $\mathcal{L}_c$-formulas, the set $\mathrm{acl}_{\Phi,c}$ is Σ^0_1 in ACL^c_0.

Proof Sketch.

Let $A \subseteq \mathcal{M}_c$ be a finite set. Note that $b \in \mathrm{acl}_{\Phi,c}(A)$ if and only if there is a finite sequence $b_0, \ldots, b_{n-1} \in \mathcal{M}_c$ where $b = b_{n-1}$ such that for each $i < n$, there exists a formula $\varphi_i(\overline{x}; \overline{y}) \in \Phi$, a tuple $\overline{a}_i$ with entries from $A \cup \{b_j\}_{j<i}$, and a tuple $\mathbf{d}_i \in \mathcal{M}_c$ satisfying

- $(\varphi_i(\overline{x}; \overline{y}), \overline{a}_i) \in \mathrm{ACL}^c_n$,
- $\mathcal{M}_c \models \varphi_i(\overline{a}_i; \mathbf{d}_i)$, and
- $b_i \in \mathbf{d}_i$.

Hence, uniformly in c, the set $\mathrm{acl}_{\Phi,c}$ is Σ^0_1 in ACL^c_n. □

Relationships Between DCL^c and $\mathrm{dcl}_{\Phi,c}$

Proposition

Uniformly in the parameter $c \in \mathrm{CompStr}$ and an encoding of a computable set Φ of Σ_n first-order $\mathcal{L}_c$-formulas, the set $\mathrm{dcl}_{\Phi,c}$ is Σ^0_1 in DCL^c_0.

Proof.

Let $A \subseteq \mathcal{M}_c$ be a finite set. Note that $b \in \mathrm{dcl}_{\Phi,c}(A)$ if and only if there is a finite sequence $b_0, \ldots, b_{n-1} \in \mathcal{M}_c$ where $b = b_{n-1}$ such that for each $i < n$, there exists a formula $\varphi_i(\overline{x}; \overline{y}) \in \Phi$, a tuple $\overline{a}_i$ with entries from $A \cup \{b_j\}_{j<i}$, and a tuple $\mathbf{d}_i \in \mathcal{M}_c$ satisfying

- $(\varphi_i(\overline{x}; \overline{y}), \overline{a}_i) \in \mathrm{DCL}^c_n$,
- $\mathcal{M}_c \models \varphi_i(\overline{a}_i; \mathbf{d}_i)$, and
- $b_i \in \mathbf{d}_i$.

Hence, uniformly in c, the set $\mathrm{dcl}_{\Phi,c}$ is Σ^0_1 in DCL^c_n. □

Bounds For Quantifier-Free Formulas

Upper bounds for quantifier-free formulas

There are straightforward upper bounds on the complexity of ACL_0^c and DCL_0^c for $c \in \mathrm{CompStr}$:

Proposition

Uniformly in $c \in \mathrm{CompStr}$, the set ACL_0^c is a Σ_2^0 class.

Corollary

Uniformly in $c \in \mathrm{CompStr}$ and in a computable collection Φ of quantifier-free $\mathcal{L}_c$-formulas, $\mathrm{acl}_{\Phi,c}$ is a Σ_2^0-class.

Proposition

Uniformly in $c \in \mathrm{CompStr}$, the set DCL_0^c is the intersection of a Π_1^0 and a Σ_1^0 class (in particular, it is a Δ_2^0 class).

As a consequence, DCL_0^c is computable from $\mathbf{0}'$.

Corollary

Uniformly in $c \in \mathrm{CompStr}$ and in a computable collection Φ of quantifier-free $\mathcal{L}_c$-formulas, $\mathrm{dcl}_{\Phi,c}$ is a Σ_2^0-class.

Lower bounds for quantifier-free formulas: ACL_0

These upper bounds are tight — further, via structures that have nice model-theoretic properties:

Proposition

There is a parameter $c \in \mathrm{CompStr}$ such that the following hold.

(a) *$\mathcal{L}_c$ has no relation symbols, i.e., $\mathcal{L}_c$ consists only of sorts.*

(b) *For each ordinal α, the theory T_c has $(|\alpha + 1|^{\omega})$-many models of size $\aleph_\alpha$. In particular, T_c is $\aleph_0$-categorical.*

(c) $\mathrm{ACL}_0^c \equiv_1 \mathrm{Fin}$. *In particular,* ACL_0^c *is a Σ^0_2-complete set.*

Lower bounds for quantifier-free formulas: ACL_0

Proof.

Let $((e_i, n_i))_{i\in\mathbb{N}}$ be a computable enumeration without repetition of $\{(e, n) : e, n \in \mathbb{N} \text{ and } \{e\}(n)\downarrow\}$.

Let $c \in$ CompStr be such that

- $\mathcal{L}_c$ consists of infinitely many sorts $(X_i)_{i\in\mathbb{N}}$ and no relation symbols,
- the underlying set of $\mathcal{M}_c$ is $\mathbb{N}$, and
- for each $i \in \mathbb{N}$, the element i is of sort X_{e_i} in $\mathcal{M}_c$.

A model of T_c is determined up to isomorphism by the number of elements in the instantiation of each sort.

ACL_0^c is 1-equivalent to $\{e : (X_e)^{\mathcal{M}_c} \text{ is finite}\}$. □

Lower bounds for quantifier-free formulas: DCL_0

Proposition

There is a parameter $c \in \mathsf{CompStr}$ *such that the following hold.*

(a) *The language* $\mathcal{L}_c$ *has one sort and a single binary relation symbol* E.

(b) *The structure* $\mathcal{M}_c$ *is a countable saturated model of* T_c *with underlying set* $\mathbb{N}$.

(c) *For each ordinal* α, *the theory* T_c *has* $(|\alpha + \omega|)$*-many models of size* $\aleph_\alpha$, *and has finite Morley rank.*

(d) *There is a computable array* $\left(U_{k,\ell}\right)_{k,\ell \in \mathbb{N}}$ *of subsets of* $\mathbb{N}$ *such that every countable model of* T_c *is isomorphic to the restriction of* $\mathcal{M}_c$ *to underlying set* $U_{k,\ell}$ *for exactly one pair* (k, ℓ).

(e) *The set* $\{a : (E(x; y), a) \in \mathsf{DCL}_0^c\}$ *has Turing degree* $\mathbf{0}'$.

Lower bounds for quantifier-free formulas: acl and dcl

Proposition

There is an $a \in \mathrm{CompStr}$ *and a computable set* Ξ *of quantifier-free first-order* $\mathcal{L}_a$*-formulas such that we can compute* Fin *from* $\mathrm{acl}_{\Xi,a}$ *via a* 1*-reduction.*

In particular, the set $\mathrm{acl}_{\Xi,a}$ *is* Σ^0_2*-complete.*

Proposition

There is a parameter $d \in \mathrm{CompStr}$ *such that* $\mathcal{L}_d$ *contains a ternary relation symbol* F *and, letting* $\Gamma := \{F(x, y; z)\}$*, we can compute* Fin *from* $\mathrm{dcl}_{\Gamma,d}$ *via a* 1*-reduction.*

In particular, the set $\mathrm{dcl}_{\Gamma,d}$ *is* Σ^0_2*-complete.*

Relationship between ACL_0, DCL_0 and acl, dcl.

We have upper bounds on the difficulty of computing $\mathrm{acl}_{\Phi,c}$ from ACL_0^c, and of computing $\mathrm{dcl}_{\Phi,c}$ from DCL_0^c, for Φ a computable set of quantifier-free first-order $\mathcal{L}_c$-formulas.

In general though, merely knowing $\mathrm{acl}_{\Phi,c}$ and $\mathrm{dcl}_{\Phi,c}$ does not lower the difficulty of computing even the Φ-fiber of ACL_0^c or DCL_0^c.

Relationship between ACL_0, DCL_0 and acl, dcl.

Proposition

There are $c_0, c_1 \in \mathrm{CompStr}$ *such that the following hold.*

(a) *The (one-sorted) language* $\mathcal{L}_{c_0} = \mathcal{L}_{c_1}$ *contains a ternary relation symbol* F *and a unary relation symbol* U.

(b) $\mathcal{M}_{c_0}$ *and* $\mathcal{M}_{c_1}$ *have the same underlying set* M *and agree on all unary relations.*

(c) *Let* $\psi(x, y, z) := F(x, y, z) \wedge \neg F(x, z, y)$, *and write* $\Psi = \{\psi(x, y; z)\}$. *For any* $A \subseteq M$,

$$\mathrm{acl}_{\Psi, c_0}(A) = \mathrm{dcl}_{\Psi, c_1}(A) = \begin{cases} M & \text{if } A \cap U \neq \emptyset, \text{ and} \\ \emptyset & \text{if } A \cap U = \emptyset. \end{cases}$$

(d) *The set* Fin *is* 1*-reducible to* $\mathrm{ACL}_0^{c_0}$, *and so* $\mathrm{ACL}_0^{c_0}$ *is a* Σ_2^0*-complete set.*

(e) $\mathrm{DCL}_0^{c_1}$ *is Turing equivalent to* $\mathbf{0}'$.

Full Bounds

Computable Morleyization

Lemma

Let $\mathcal{L}$ be a computable language and $\mathcal{A}$ a computable $\mathcal{L}$-structure. For each $n \in \mathbb{N}$ there is a computable language $\mathcal{L}_n$ and a $\mathbf{0}^{(n)}$-computable $\mathcal{L}_n$-structure $\mathcal{A}_n$ such that

- $\mathcal{L} \subseteq \mathcal{L}_n \subseteq \mathcal{L}_{n+1}$,
- *$\mathcal{A}$ is the reduct of $\mathcal{A}_n$ to the language $\mathcal{L}$,*
- *for each first-order $\mathcal{L}_n$-formula φ there is a first-order $\mathcal{L}$-formula ψ_φ (of the same type as φ) such that*

  $$\mathcal{A}_n \models (\forall x_0, \dots, x_{k-1})\ \varphi(x_0, \dots, x_{k-1}) \leftrightarrow \psi_\varphi(x_0, \dots, x_{k-1}),$$

  where k is the number of free variables of φ, and
- *for each first-order $\mathcal{L}$-formula ψ that is a Boolean combination of Σ_n-formulas, there is a first-order quantifier-free $\mathcal{L}_n$-formula φ_ψ (of the same type as ψ) such that*

  $$\mathcal{A}_n \models (\forall x_0, \dots, x_{k-1})\ \psi(x_0, \dots, x_{k-1}) \leftrightarrow \varphi_\psi(x_0, \dots, x_{k-1}),$$

  where k is the number of free variables of ψ.

Upper bounds for Boolean combinations of Σ_n-formulas

Let $n \in \mathbb{N}$. Uniformly in $c \in \mathrm{CompStr}$, we have that

- ACL_n^c is a Σ^0_{n+2} set, and
- DCL_n^c is a Δ^0_{n+2} set.

Further, uniformly in $c \in \mathrm{CompStr}$ and in a computable collection Φ of first-order $\mathcal{L}_c$-formulas of quantifier rank at most n, we have that

- $\mathrm{acl}_{\Phi,c}$ is a Σ^0_{n+2} set, and
- $\mathrm{dcl}_{\Phi,c}$ is a Σ^0_{n+2} set.

Proof.

By the computable Morleyization, ACL_n is equivalent to the relativization of ACL_0 to the class of structures computable in $\mathbf{0}^{(n)}$, and DCL_n is equivalent to the relativization of DCL_0 to the class of structures computable in $\mathbf{0}^{(n)}$.

Therefore by the quantifier-free upper bounds, ACL_n^c is a $\Sigma^0_2(\mathbf{0}^{(n)})$ class and DCL_n^c is a $\Delta^0_2(\mathbf{0}^{(n)})$ class. As $\mathrm{acl}_{\Phi,c}$ is Σ^0_1 in ACL_n^c we have $\mathrm{acl}_{\Phi,c}$ is also a $\Sigma^0_2(\mathbf{0}^{(n)})$ set. As $\mathrm{dcl}_{\Phi,c}$ is Σ^0_1 in DCL_n^c we have $\mathrm{dcl}_{\Phi,c}$ is also a $\Sigma^0_2(\mathbf{0}^{(n)})$ set. $\square$

Directed $\mathbb{N}$-chains

Let $\mathcal{L}$ be a language containing a sort N and a relation symbol S of type $N \times N$. Let $\mathcal{A}$ be an $\mathcal{L}$-structure.

Call $(N^{\mathcal{A}}, S^{\mathcal{A}})$ a **directed $\mathbb{N}$-chain** when it is isomorphic to a single-sorted structure with underlying set $\mathbb{N}$ and language $\{S\}$, in which $S(k, \ell)$ holds precisely when $\ell = k + 1$.

In other words, $(N^{\mathcal{A}}, S^{\mathcal{A}})$ is a directed $\mathbb{N}$-chain if there is a (necessarily unique) isomorphism between it and $\mathbb{N}$ with its successor function viewed as a directed graph; write $\widehat{\ell}$ to denote the corresponding element of $N^{\mathcal{A}}$.

Directed $\mathbb{N}$-chains

Lemma

Let $\mathcal{L}$ be a language containing a sort N and a relation symbol S of type $N \times N$ (and possibly other sorts and relation symbols). Let $\mathcal{A}$ be an $\mathcal{L}$-structure such that $(N^{\mathcal{A}}, S^{\mathcal{A}})$ is a directed $\mathbb{N}$-chain. Let $k \in \mathbb{N}$ and let $h(\overline{x}, m)$ be an $\mathcal{L}$-formula that is a Boolean combination of Σ_k-formulas, where $\overline{x}$ is of some type X, and m has sort N. Suppose that

$$\mathcal{A} \models (\forall \overline{x} : X)(\exists^{\leq 1} m : N)(\exists p : N)\ S(m, p) \wedge \big(h(\overline{x}, m) \leftrightarrow \neg h(\overline{x}, p)\big).$$

Let $H \colon X^{\mathcal{A}} \times \mathbb{N} \to \{\mathrm{True}, \mathrm{False}\}$ be the function where $H(\overline{a}, \ell) = \mathrm{True}$ if and only if $\mathcal{A} \models h(\overline{a}, \widehat{\ell})$. Note that $\lim_{\ell \to \infty} H(\overline{a}, \ell)$ exists for all $\overline{a} \in X^{\mathcal{A}}$. There is an $\mathcal{L}$-formula $h'(\overline{x})$, where $\overline{x}$ is of type X, such that h' is a Boolean combination of Σ_{k+1}-formulas and for all $\overline{a} \in X^{\mathcal{A}}$,

$$\mathcal{A} \models h'(\overline{a}) \quad \text{if and only if} \quad \lim_{m \to \infty} H(\overline{a}, m) = \mathrm{True}.$$

Lower bounds for Boolean combinations of Σ_n-formulas

Proposition

Let $n \in \mathbb{N}$ and let $\mathcal{L}$ be a language containing a sort N and a relation symbol S of type $N \times N$ (and possibly other sorts and relation symbols). Suppose $\mathcal{A}$ is an $\mathcal{L}$-structure that is computable in $\mathbf{0}^{(n)}$ and such that $(N^{\mathcal{A}}, S^{\mathcal{A}})$ is a computable directed $\mathbb{N}$-chain. Then there is a computable language $\mathcal{L}^+$ and a computable $\mathcal{L}^+$-structure $\mathcal{A}^+$ such that for every relation symbol $R \in \mathcal{L}$ other than S, there is an $\mathcal{L}^+$-formula φ_R that is a Boolean combination of Σ_n-formulas for which $R^{\mathcal{A}} = (\varphi_R)^{\mathcal{A}^+}$.

Proof.

Define $\mathcal{L}^+$ to have the same sorts as $\mathcal{L}$, and such that for each relation symbol $R \in \mathcal{L}$ other than S, there is a relation symbol $R^+ \in \mathcal{L}^+$ of type $X \times N^n$, where X is the type of R.

For each $R \in \mathcal{L}$ other than S, each tuple $\overline{a} \in X^{\mathcal{A}^+}$ where X is the type of R, and any $\ell_0, \ldots, \ell_{n-1} \in \mathbb{N}$, code n-fold limits of a computable function into whether $\mathcal{A}^+ \models R^+(\overline{a}, \widehat{\ell_0}, \ldots, \widehat{\ell_{n-1}})$ holds.

Apply the Lemma repeatedly (n times) to obtain the desired formula. $\square$

Lower bounds for Boolean combinations of Σ_n-formulas

Theorem

For each $n \in \mathbb{N}$, the following hold.

(a) *There exists $a \in \mathrm{CompStr}$ such that ACL_n^a is a Σ^0_{n+2}-complete set.*

(b) *There exists $b \in \mathrm{CompStr}$ such that $\mathrm{DCL}_n^b \equiv_{\mathrm{T}} \mathbf{0}^{(n+1)}$.*

(c) *There exists $c \in \mathrm{CompStr}$ and a computable set Φ of first-order $\mathcal{L}_c$-formulas, all of quantifier rank at most n such that $\mathrm{acl}_{\Phi,c}$ is a Σ^0_{n+2}-complete set.*

(d) *There exists $d \in \mathrm{CompStr}$ and a computable set Θ of first-order $\mathcal{L}_d$-formulas, all of quantifier rank at most n, such that $\mathrm{dcl}_{\Theta,d}$ is a Σ^0_{n+2}-complete set.*

Lower bounds for Boolean combinations of Σ_n-formulas

Proof Sketch.

Let $\mathcal{P}$ be the structure constructed in the proof of the quantifier-free lower bound on ACL, relativized to the oracle $\mathbf{0}^{(n)}$, i.e., so that $\mathcal{P}$ is computable from $\mathbf{0}^{(n)}$. Let the structure $\mathcal{P}^*$ be $\mathcal{P}$ augmented with a sort N (instantiated on a new set of elements) along with a relation symbol S of type $N \times N$, such that $(N^{\mathcal{P}^*}, S^{\mathcal{P}^*})$ is a computable directed $\mathbb{N}$-chain.

Part (a) then follows by applying the previous proposition to $\mathcal{P}^*$ to obtain some computable structure, namely $\mathcal{M}_a$ for some $a \in \mathrm{CompStr}$. Then ACL_n^a is a $\Sigma_2^0(\mathbf{0}^{(n)})$-complete set.

Parts (b) - (d) are similar. □

Thank You!