The Computability of the Tree Antichain Theorem

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Computable Rings

Definition

A *computable ring* is a computable subset $A \subseteq \mathbb{N}$ equipped with two computable binary operations $+$ and $\cdot$ on A, together with elements $0, 1 \in A$ such that $R = (A, 0, 1, +, \cdot)$ is a ring.

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All rings will be *countable*, *commutative*, with an *identity element*.

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P is a *minimal prime ideal* if it does not properly contain a prime ideal.

Noether's Primary Decomposition Lemma

Primary Decomposition Lemma

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Primary Decomposition Lemma

If R contains infinitely many minimal prime ideals, then R is not Noetherian, i.e. R contains an infinite strictly ascending chain of ideals

$$I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots \subset R, \ \ n \in \mathbb{N}.$$

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Let $I_0 = \langle 0 \rangle_R \subset R$.
Since R contains infinitely many minimal primes, $\langle 0 \rangle_R \subset R$ is not a prime ideal. Therefore there exist $a_1, b_1 \in R$ such that $a_1, b_1 \notin I_0$ but $a_1 b_1 = 0 \in I_0$. Now, either a_1 or b_1 is contained in infinitely many minimal primes; add it to I_0 to get $I_1 \supset I_0$.

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Repeat with the invariant that

$$I_k = \langle c_1, c_2, \cdots, c_k \rangle_R \subset R, \ k \in \mathbb{N},$$

is contained in infinitely many minimal primes, and therefore is not prime itself. Uses $\emptyset''$.

The "Big Five:"

- RCA_0 : Recursive Comprehension Axiom
- WKL_0 : Weak König's Lemma
- ACA_0 : Arithmetic Comprehension Axiom
- ATR_0 : Arithmetic Transfinite Recursion
- $\Pi^1_1-\mathsf{CA}_0$: Π^1_1-Comprehension Axiom

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- AMT : Atomic Model Theorem

The Tree Antichain Theorem

Definition

Let $T \subseteq 2^{<\mathbb{N}}$ be a tree. We say that T is completely branching if for all $\sigma \in T$, $\sigma^+ = \{\sigma 0, \sigma 1\} \subset 2^{<\mathbb{N}}$, either

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TAC (Tree Antichain Theorem–Equivalent Version)

Every infinite tree $T \subseteq 2^{<\mathbb{N}}$ with no terminal nodes and infinitely many splittings has an infinite antichain.

Two Paths to TAC

Fact (RCA_0)

TAC follows from each of 2-MLR and ADS (individually).

Fact (RCA_0)

TAC is restricted Π^1_2.

Fact (RCA_0)

TAC does not follow from WKL

Corollary

TAC is not equivalent to any other "known" subsystem of Second-Order Arithmetic.

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- We say that ideals $I, J \subseteq R$ are <u>uniformly coprime</u> if for all $x \in I \cap J$ there exist $y \in I$, $z \in J$, and $a, b \in R$ such that

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Primary Decomposition for Restricted Classes of Rings

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Theorem A

If R has infinitely many coprime minimal primes, then R is not Noetherian.

Theorem B

If R has infinitely many uniformly coprime minimal primes, then R is not Noetherian.

Algebraic Characterizations of TAC

Theorem ($\mathsf{RCA}_0 + \mathsf{B}\Sigma_2$)

Theorem B is equivalent to TAC.

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Theorem ($RCA_0 + B\Sigma_2$)

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Conjecture ($RCA_0 + B\Sigma_2$)

Theorem A is equivalent to TAC.

TAC implies Theorem B

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- every $\sigma \in T$ corresponds to some (zero-divisor) $x_\sigma \in R$;

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If $\{\alpha_i : i \in \mathbb{N}\}$ is an infinite $T-$antichain, and

$$I_N = Ann(\prod_{i=1}^{N} x_{\alpha_i}),$$

then

$$I_0 \subset I_1 \subset I_2 \cdots \subset I_N \subset \cdots .$$

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- R is a PIR; every ideal $I \subset R$ is generated by a monomial.
- Given an infinite strictly ascending $R-$chain, one can effectively find a principle generator for each ideal in the chain and use $B\Sigma_2$ along with the sequence of exponents of these generators to build an infinite antichain of T in the context.

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Definition (RCA_0)

For each $n \in \mathbb{N}$, let $n-$TAC be the principle that says "for every infinite tree $T \subseteq 2^{<\mathbb{N}}$ with infinitely many splittings, there is a (path-)nonincreasing $f_T : T \to \mathbb{N}$ such that:

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TAC $\longrightarrow$ Theorem B $\longrightarrow$ WTAC, over RCA_0.
TAC $\longleftrightarrow$ Theorem A/B $\longleftrightarrow$ WTAC, over $\mathsf{RCA}_0 + \mathsf{B}\Sigma_2$.

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Theorem

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Question (RCA_0)

- *Does PDL follow from some measure-theoretic principle (randomness)?*
- Π^1_1*−conservativity for (n−)TAC, PDL?*

Thank You!