Ramsey theory on infinite structures

Natasha Dobrinen

University of Notre Dame

Computability and Combinatorics 2023

University of Connecticut, May 15–21, 2023

Research supported by NSF grant DMS-1901753

Ramsey's Theorems

Theorem (Finite Ramsey Theorem)

Given $m < n$ *and* $2 \leq r$, *there is a* p *large enough so that for any coloring* $\chi : [p]^m \to r$, *there is an* $N \in [p]^n$ *for which* $\chi \restriction [N]^m$ *takes only one color.*

Theorem (Infinite Ramsey Theorem)

Given m, r *and a coloring* $\chi : [\omega]^m \to r$, *there is an infinite subset* $N \in [\omega]^\omega$ *for which* $\chi \restriction [N]^m$ *takes only one color.*

Finite Structural Ramsey Theory

For structures $\mathbf{A}, \mathbf{B}$, write $\mathbf{A} \leq \mathbf{B}$ iff $\mathbf{A}$ embeds into $\mathbf{B}$.

$\binom{\mathbf{B}}{\mathbf{A}}$ denotes the set of all copies of $\mathbf{A}$ in $\mathbf{B}$.

A class $\mathcal{K}$ of finite structures has the **Ramsey Property** if given $\mathbf{A} \leq \mathbf{B}$ in $\mathcal{K}$ and r, there is $\mathbf{C} \in \mathcal{K}$ so that

$$\forall \chi : \binom{\mathbf{C}}{\mathbf{A}} \to r \quad \exists \mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}},\ \chi \restriction \binom{\mathbf{B}'}{\mathbf{A}} \text{ is constant.}$$

Lots of work done! (e.g., Nešetřil–Rödl, Hubička–Nešetřil)

Examples: The classes of finite linear orders, finite ordered graphs, and finite ordered k-clique-free graphs have RP.

Which infinite structures carry
analogues of the Infinite Ramsey Theorem?

Example: The Rationals as a Dense Linear Order

- $(\mathbb{Q}, <)$ has a Pigeonhole Principle.

Example: The Rationals as a Dense Linear Order

- $(\mathbb{Q}, <)$ has a Pigeonhole Principle.
- Ramsey's Theorem fails for pairs of rationals. (Sierpiński, 1933)

Key Idea: Enumerate $\mathbb{Q}$ as $\langle q_0, q_1, q_2, \ldots \rangle$

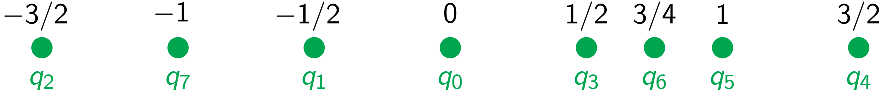
$$\text{Define a coloring : for } i < j,\ c(\{q_i, q_j\}) = \begin{cases} \text{red} & \text{if } q_i < q_j \\ \text{blue} & \text{if } q_j < q_i \end{cases}$$

Example: The Rationals as a Dense Linear Order

- $(\mathbb{Q}, <)$ has a Pigeonhole Principle.
- Ramsey's Theorem fails for pairs of rationals. (Sierpiński, 1933)

Key Idea: Enumerate $\mathbb{Q}$ as $\langle q_0, q_1, q_2, \ldots \rangle$

$$\text{Define a coloring : for } i < j,\ c(\{q_i, q_j\}) = \begin{cases} \text{red} & \text{if } q_i < q_j \\ \text{blue} & \text{if } q_j < q_i \end{cases}$$



These patterns are unavoidable.

Example: The Rationals as a Dense Linear Order

- $(\mathbb{Q}, <)$ has a Pigeonhole Principle.
- Ramsey's Theorem fails for pairs of rationals. (Sierpiński, 1933)

Key Idea: Enumerate $\mathbb{Q}$ as $\langle q_0, q_1, q_2, \ldots \rangle$

$$\text{Define a coloring : for } i < j,\ c(\{q_i, q_j\}) = \begin{cases} \text{red} & \text{if } q_i < q_j \\ \text{blue} & \text{if } q_j < q_i \end{cases}$$

$-3/2$	-1	$-1/2$	0	$1/2$	$3/4$	1	$3/2$
●	●	●	●	●	●	●	●
q_2	q_7	q_1	q_0	q_3	q_6	q_5	q_4

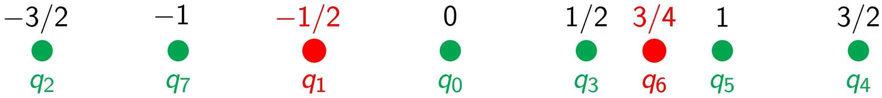
These patterns are unavoidable.

Example: The Rationals as a Dense Linear Order

- $(\mathbb{Q}, <)$ has a Pigeonhole Principle.
- Ramsey's Theorem fails for pairs of rationals. (Sierpiński, 1933)

Key Idea: Enumerate $\mathbb{Q}$ as $\langle q_0, q_1, q_2, \ldots \rangle$

$$\text{Define a coloring : for } i < j,\ c(\{q_i, q_j\}) = \begin{cases} \text{red} & \text{if } q_i < q_j \\ \text{blue} & \text{if } q_j < q_i \end{cases}$$



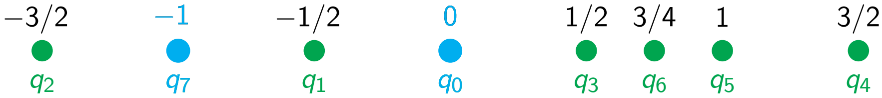
These patterns are unavoidable.

Example: The Rationals as a Dense Linear Order

- $(\mathbb{Q}, <)$ has a Pigeonhole Principle.
- Ramsey's Theorem fails for pairs of rationals. (Sierpiński, 1933)

Key Idea: Enumerate $\mathbb{Q}$ as $\langle q_0, q_1, q_2, \ldots \rangle$

$$\text{Define a coloring : for } i < j,\ c(\{q_i, q_j\}) = \begin{cases} \text{red} & \text{if } q_i < q_j \\ \text{blue} & \text{if } q_j < q_i \end{cases}$$



These patterns are unavoidable.

Ramsey Theory on the Rationals

Theorem (D. Devlin, 1979)

Given m, if the m-element subsets of $\mathbb{Q}$ are colored by finitely many colors, then there is a subcopy $\mathbb{Q}' \subseteq \mathbb{Q}$ forming a dense linear order in which the m-element subsets take no more than $C_{2m-1}(2m-1)!$ colors. This bound is optimal.

m	**Bound**
1	1
2	2
3	16
4	272

C_i is from

$\tan(x) = \sum_{i=0}^{\infty} C_i x^i$

- Galvin (1968) The bound for pairs is two.
- Laver (1969) Upper bounds for all finite sets.

Infinite Structural Ramsey Theory

Let $\mathbf{K}$ be an infinite structure.

$\mathbf{K}$ has **finite big Ramsey degrees** if for each finite $\mathbf{A} \leq \mathbf{K}$, $\exists T$ such that $\forall r$, $\forall\ \chi : \binom{\mathbf{K}}{\mathbf{A}} \to r$, $\exists \mathbf{K}' \in \binom{\mathbf{K}}{\mathbf{K}}$ such that $|\chi \restriction \binom{\mathbf{K}'}{\mathbf{A}}| \leq T$.

The **big Ramsey degree** of $\mathbf{A}$ in $\mathbf{K}$, $T(\mathbf{A})$, is the least such T.

Infinite Structural Ramsey Theory

Let $\mathbf{K}$ be an infinite structure.

$\mathbf{K}$ has **finite big Ramsey degrees** if for each finite $\mathbf{A} \leq \mathbf{K}$, $\exists T$ such that $\forall r$, $\forall\ \chi : \binom{\mathbf{K}}{\mathbf{A}} \to r$, $\exists \mathbf{K}' \in \binom{\mathbf{K}}{\mathbf{K}}$ such that $|\chi \restriction \binom{\mathbf{K}'}{\mathbf{A}}| \leq T$.

The **big Ramsey degree** of $\mathbf{A}$ in $\mathbf{K}$, $T(\mathbf{A})$, is the least such T.

Let $\mathcal{K}$ be a Fraïssé class with limit $\mathbf{K}$.

- If $|\mathrm{Aut}(\mathbf{K})| > 1$, then $\exists \mathbf{A} \in \mathcal{K}$ with $T(\mathbf{A}) > 1$ then $\exists \mathbf{A} \in \mathcal{K}$ with $T(\mathbf{A}) > 1$ (or infinite). (Hjorth 2008)

Topological Dynamics and Ramsey Theory

Theorem (Kechris–Pestov–Todorcevic, 2005)

A Fraïssé class $\mathcal{K}$ of finite structures has the Ramsey property if and only if $Aut(\mathbf{K})$ is extremely amenable, where $\mathbf{K}$ is the homogeneous structure universal for $\mathcal{K}$.

Theorem (Zucker, 2019)

If $\mathbf{K}$ has a big Ramsey structure, then $Aut(\mathbf{K})$ admits a unique universal completion flow.

Homogeneous and Universal Structures

A structure **K** is **homogeneous** if every isomorphism between two finite induced substructures of **K** extends to an automorphism of **K**.

A structure **K** is **universal** for a class of structures $\mathcal{K}$ if every structure in $\mathcal{K}$ embeds into **K**.

Homogeneous structures include

- $(\mathbb{Q}, <)$ The rationals
- $(\mathcal{R}, E)$ The Rado graph
- $(\mathcal{H}_3, E)$ The triangle-free Henson graph

Homogeneous structures are good environments for Ramsey theory.

Fraïssé correspondence.

Big Ramsey Degree results, a sampling

- 1933. $T(\text{Pairs}, \mathbb{Q}) \geq 2$. (Sierpiński)
- 1975. $T(\text{Edge}, \mathcal{R}) \geq 2$. (Erdős, Hajnal, Pósa)
- 1979. $(\mathbb{Q}, <)$: All BRD computed. (D. Devlin)
- 1986. $T(\text{Vertex}, \mathcal{H}_3) = 1$. (Komjáth, Rödl)
- 1989. $T(\text{Vertex}, \mathcal{H}_n) = 1$. (El-Zahar, Sauer)
- 1996. $T(\text{Edge}, \mathcal{R}) = 2$. (Pouzet, Sauer)
- 1998. $T(\text{Edge}, \mathcal{H}_3) = 2$. (Sauer)
- 2006, 2008. The Rado graph: All BRD characterized; computed. (Laflamme, Sauer, Vuksanović); (J. Larson)
- 2010. Dense Local Order **S**(2): All BRD computed. Also $\mathbb{Q}_n$. (Laflamme, Nguyen Van Thé, Sauer)

Developments via coding trees and forcing (arxiv dates)

- 2017. Triangle-free Henson graphs: Very good Upper Bounds.

Developments via coding trees and forcing (arxiv dates)

- 2017. Triangle-free Henson graphs: Very good Upper Bounds. Exact bounds via small tweak in 2020. (D.) and independently (BCHKVZ)

Developments via coding trees and forcing (arxiv dates)

- 2017. Triangle-free Henson graphs: Very good Upper Bounds. Exact bounds via small tweak in 2020. (D.) and independently (BCHKVZ)
- 2019. All Henson graphs: Good Upper Bounds. (D.)
- 2019. ∞-dimensional RT for Borel sets of Rado graphs. (D.)
- 2020. Binary rel. Forb($\mathcal{F}$): Upper Bounds. (Zucker)
- 2020. Exact BRD and indivisibility for higher arity SDAP$^+$ structures. (Coulson, D., Patel)
- 2021. Binary rel. Forb($\mathcal{F}$): Exact BRD. (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker)
- 2022. ∞-dimensional RT structures with SDAP$^+$. (recovers Exact BRD). (D.)
- 2023. ∞-dimensional RT for binary rel. Flim(Forb($\mathcal{F}$)). (recovers Exact BRD). (D., Zucker)

Recent developments not using forcing (arxiv dates)

- 2018. Certain homogeneous metric spaces and certain universal structures: Upper Bounds. (Mašulović) Category Th.
- 2019. 3-uniform hypergraphs: Upper Bounds. (Balko, Chodounský, Hubička, Konečný, Vena) Product Milliken Theorem.
- 2019. Countable ordinals and big Ramsey degrees: Upper Bounds. (Mašulović, Šobot)
- 2020. Circular directed graphs: Exact BRD Computed. (Dasilva Barbosa) Category Theory.
- 2020. Homogeneous partial order: Upper Bounds. (Hubička) RT for parameter words. First non-forcing proof for $\mathcal{H}_3$.
- 2021. Homogenous graphs with forbidden cycles (metric spaces): Upper Bounds. (Balko, Chodounský, Hubička, Konečný, Nešetřil, Vena) RT for parameter words.
- 2022. Big Ramsey degrees in ultraproducts of finite structures. (Bartošová, Džamonja, Patel, Scow)

Recent developments not using forcing (arxiv dates)

- 2023. Homogeneous partial order: Exact BRD. (BCDHKVZ)
- 2023. Big Ramsey degrees and infinite languages. (Braunfeld, Chodounský, de Rancourt, Hubička, Kawach, Konečný)
  Laver's Product Milliken Theorem for infinitely many trees.
- 2023. Type-respecting amalgamation and big Ramsey degrees. (Aranda, Braunfeld, Chodounský, Hubička, Konečný, Nešetřil, Zucker)
- 2023. Big Ramsey degrees in the metric setting. (Bice, de Rancourt, Hubička, Konečný)
- 2023. Big Ramsey degrees of Countable Ordinals. (Boyland, Gasarch, Hurtig, Rust)
- 2023. Big Ramsey degrees in the metric setting. (Bice, de Rancourt, Hubička, Konečný)
- 2023+. Certain Forb($\mathcal{F}$) binary and higher arities. (BCDKNVZ)

Characterizations of Big Ramsey Degrees

Big Ramsey Degree Characterizations

Big Ramsey degrees in an infinite binary relational structure **K** are (so far) characterized via

I. Enumerating the universe of **K** and forming the coding tree of 1-types;

II. Diagonal antichains in the coding tree;

III. Passing types;

IV. Sometimes more (e.g., FAP with forbidden substructure, generic partial order)

First Ingredient: Enumerating the Universe

Letting the universe of **K** be ω induces a coding tree of 1-types.

First Ingredient: Enumerating the Universe

Letting the universe of $\mathbf{K}$ be ω induces a coding tree of 1-types.

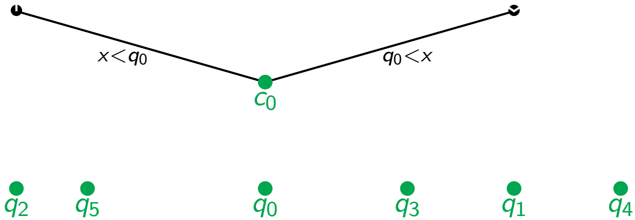
Let $\mathbf{K}$ be a homogeneous structure with finitely many relations of arity at most two and vertices $\langle v_n : n < \omega \rangle$.

Let $\mathbf{K}_n$ denote $\mathbf{K} \restriction \{v_i : i < n\}$.

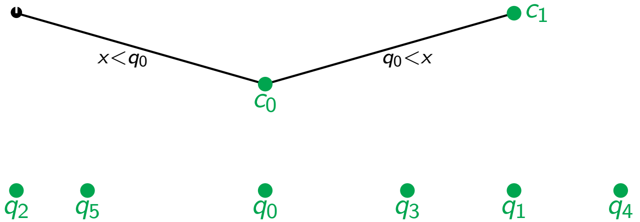
Def. (Coulson-D.-Patel) The **coding tree of 1-types** $\mathbb{S}(\mathbf{K})$ is the set of all complete 1-types over $\mathbf{K}_n$, $n < \omega$, along with a function $c : \omega \to \mathbb{S}(\mathbf{K})$ where $c(n)$ is the 1-type of v_n over $\mathbf{K}_n$. The tree-ordering is inclusion.

Coding Tree of 1-types for $(\mathbb{Q}, <)$

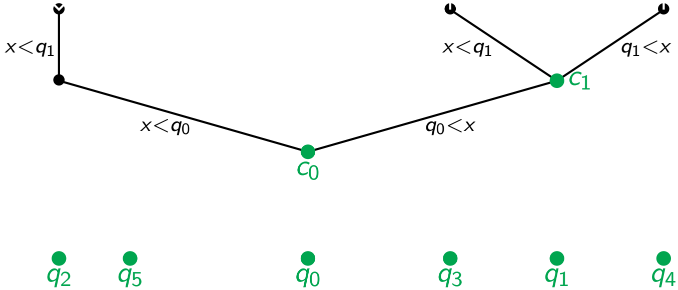
Coding Tree of 1-types for $(\mathbb{Q}, <)$



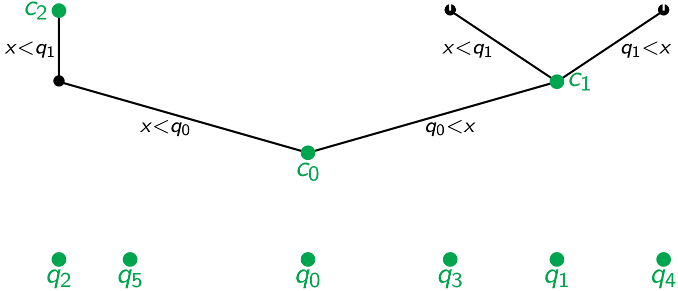
Coding Tree of 1-types for $(\mathbb{Q}, <)$



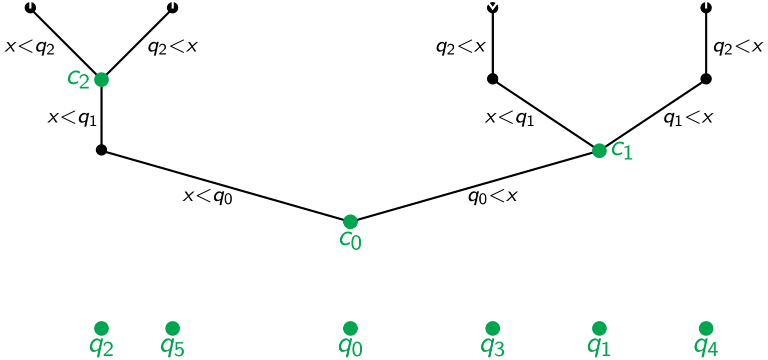
Coding Tree of 1-types for $(\mathbb{Q}, <)$



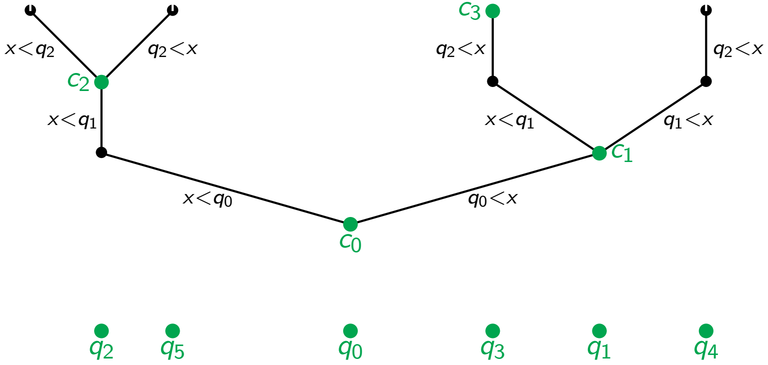
Coding Tree of 1-types for $(\mathbb{Q}, <)$



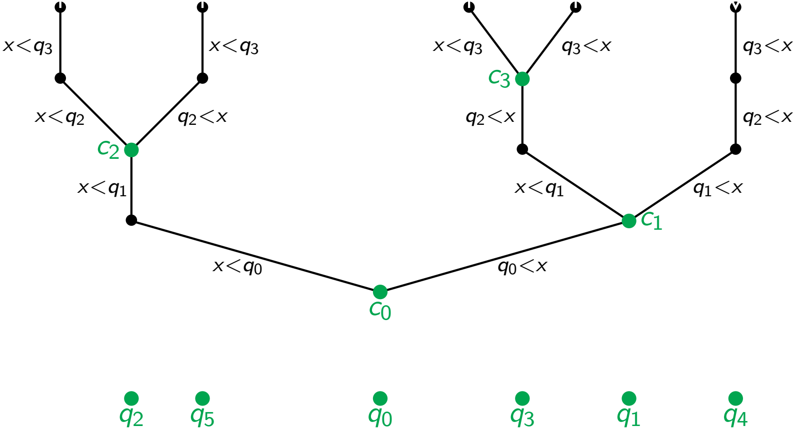
Coding Tree of 1-types for $(\mathbb{Q}, <)$



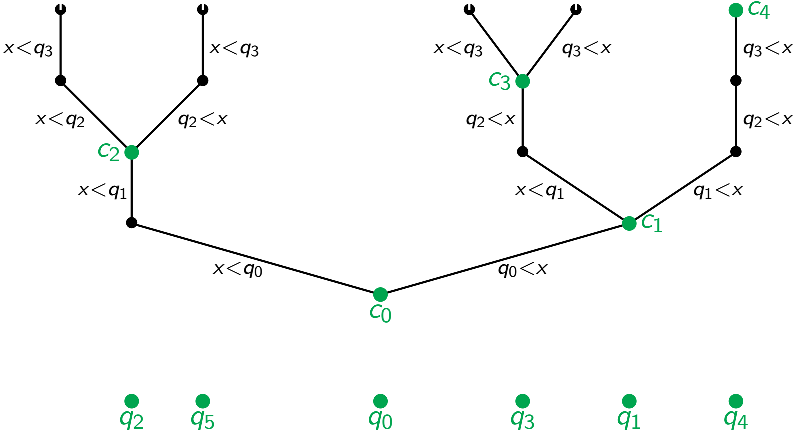
Coding Tree of 1-types for $(\mathbb{Q}, <)$



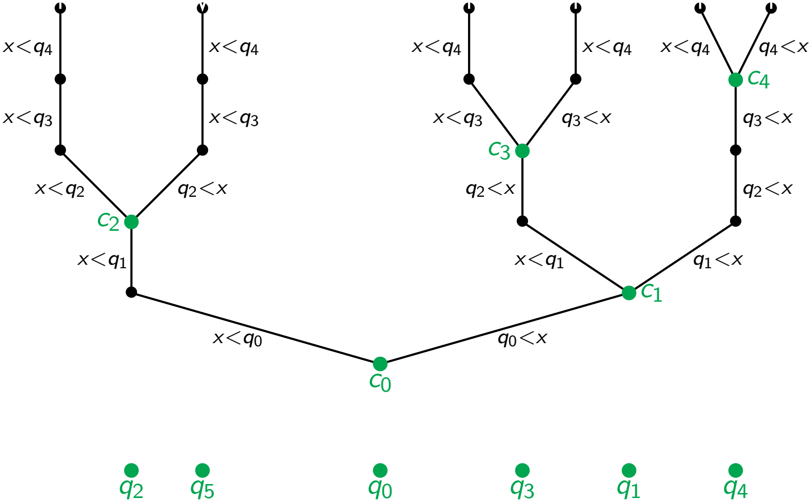
Coding Tree of 1-types for $(\mathbb{Q}, <)$



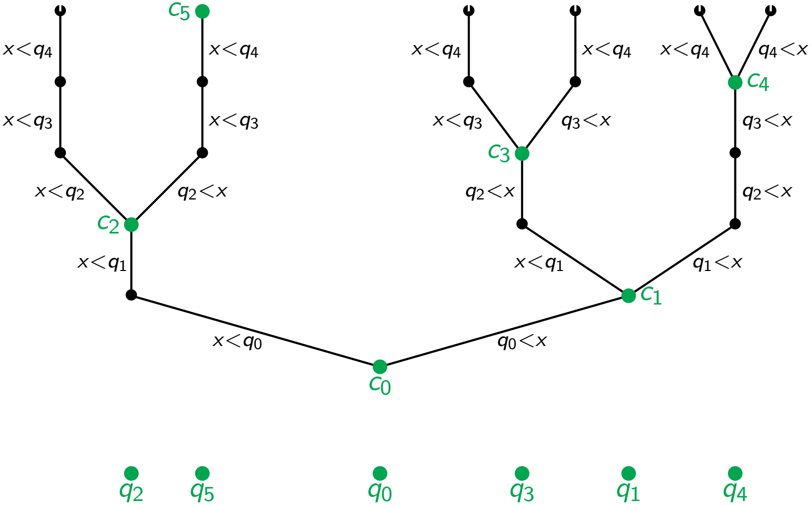
Coding Tree of 1-types for $(\mathbb{Q}, <)$



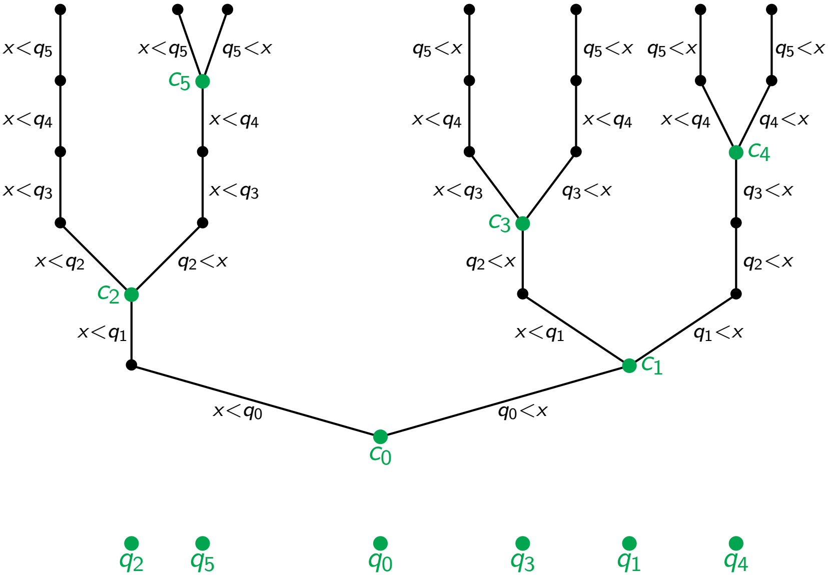
Coding Tree of 1-types for $(\mathbb{Q}, <)$



Coding Tree of 1-types for $(\mathbb{Q}, <)$

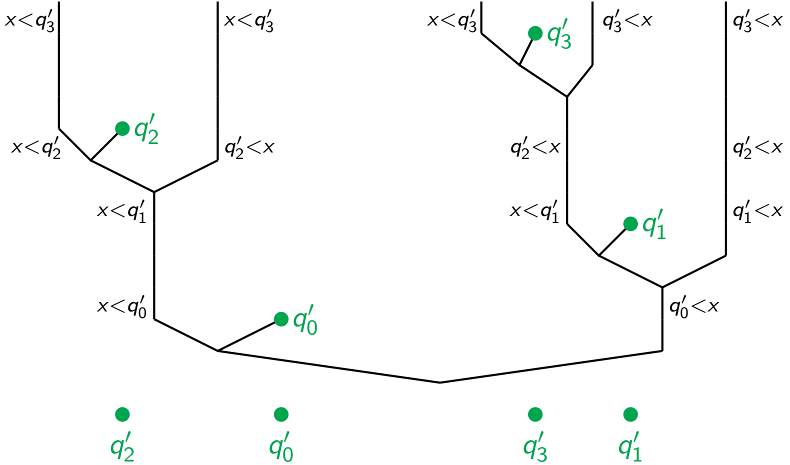


Coding Tree of 1-types for $(\mathbb{Q}, <)$



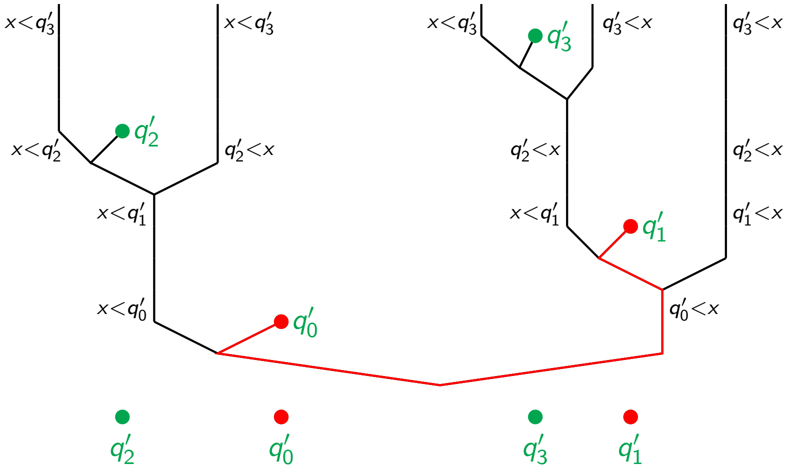
Second Ingredient: Diagonal Antichains

An antichain is diagonal any two nodes in its meet closure have different lengths.



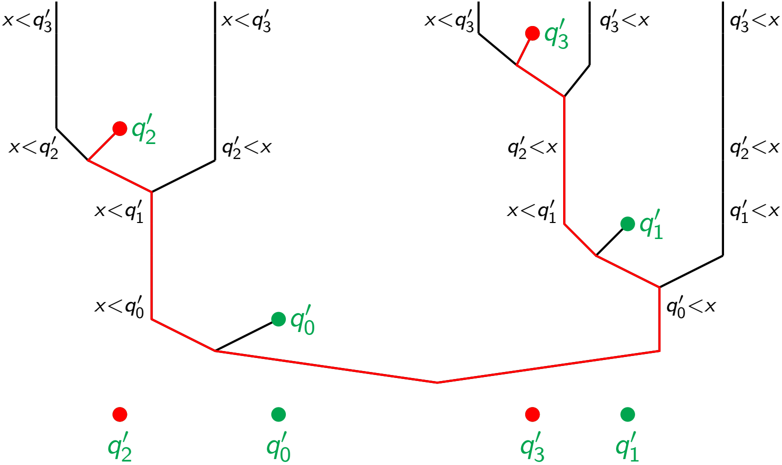
Devlin's Exact Bounds for $\mathbb{Q}$

The BRD of k-sized subsets of $\mathbb{Q}$ is exactly the number of diagonal antichains in the coding tree of 1-types of size k.

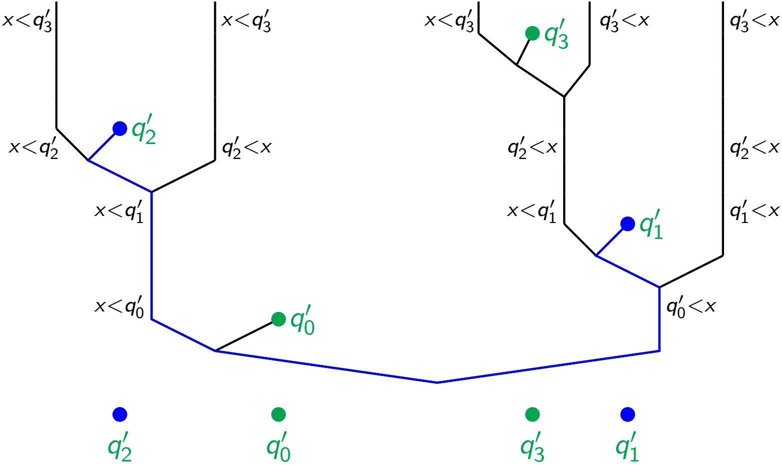


Devlin's Exact Bounds for $\mathbb{Q}$

The BRD of k-sized subsets of $\mathbb{Q}$ is exactly the number of diagonal antichains in the coding tree of 1-types of size k.

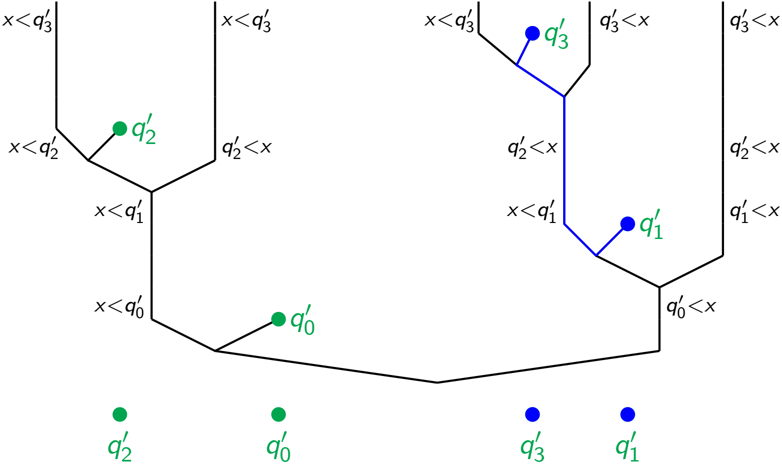


The BRD of k-sized subsets of $\mathbb{Q}$ is exactly the number of diagonal antichains in the coding tree of 1-types of size k.



Devlin's Exact Bounds for $\mathbb{Q}$

The BRD of k-sized subsets of $\mathbb{Q}$ is exactly the number of diagonal antichains in the coding tree of 1-types of size k.

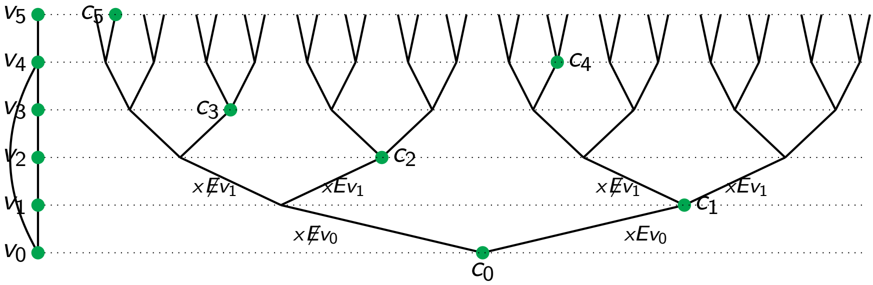


Third Ingredient for Rado graph (and others)

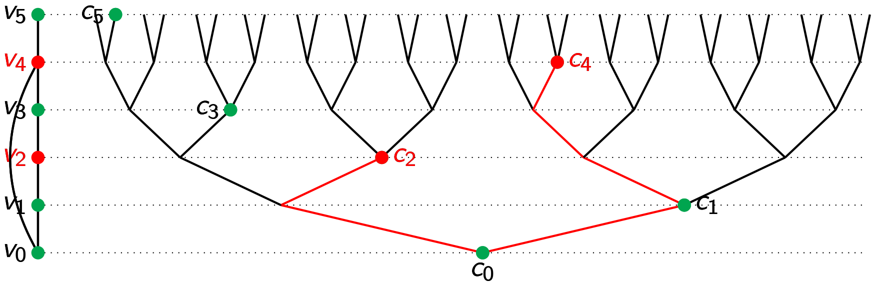
For the Rado graph, a Third Ingredient is involved in big Ramsey degrees:

passing types, the relations encoded as a longer coding node passes by a shorter one.

Coding Tree of 1-types for the Rado graph



Coding Tree of 1-types for the Rado graph



Big Ramsey degrees characterized by I–III

Theorem (Laflamme–Sauer–Vuksanovic (LSV))

The big Ramsey degrees in unrestricted structures in finitely many binary relations are characterized by diagonal antichains.

Big Ramsey degrees characterized by I–III

Theorem (Laflamme–Sauer–Vuksanovic (LSV))

The big Ramsey degrees in unrestricted structures in finitely many binary relations are characterized by diagonal antichains.

Theorem (Coulson–N.D.–Patel)

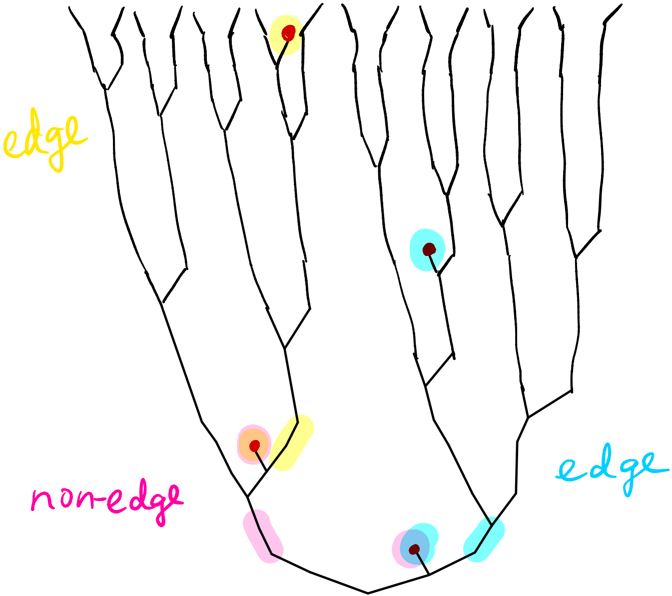
Let $\mathcal{L}$ be a finite relational language and let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit satisfying the SDAP$^+$. Let $\mathbf{K} = Flim(\mathcal{K})$.

I. $\mathbf{K}$ is indivisible.

II. If $\mathcal{L}$ has no relations of arity greater than two, then $\mathbf{K}$ has big Ramsey degrees characterized by diagonal antichains.

[CDP] recovers [Devlin], [LSV], [L–Nguyen Van Thé–S], and gives new results for $\mathbb{Q}$, $\mathbb{Q}_n$, $\mathbb{Q}_{\mathbb{Q}}$, $(\mathbb{Q}_{\mathbb{Q}})_n$, generic k-partite graph, ordered versions.

BRD in Rado Graph



Upper Bounds for Binary Free Amalgamation Classes

A structure is **irreducible** if any two vertices are in some relation; e.g., k-clique, complete tournament.

Free amalgamation classes are exactly of the form $\mathrm{Forb}(\mathcal{F})$, where $\mathcal{F}$ is a set of finite **irreducible** structures.

Upper Bounds for Binary Free Amalgamation Classes

A structure is **irreducible** if any two vertices are in some relation; e.g., k-clique, complete tournament.

Free amalgamation classes are exactly of the form Forb($\mathcal{F}$), where $\mathcal{F}$ is a set of finite **irreducible** structures.

Theorem (Zucker, 2020)

Let $\mathcal{F}$ be a finite set of finite irreducible structures in a finite relational language with relations of arity at most two. Then the homogeneous structure **K** *universal for* $\mathrm{Forb}(\mathcal{F})$ *has finite big Ramsey degrees.*

Exact Big Ramsey Degrees for binary free amalgamation

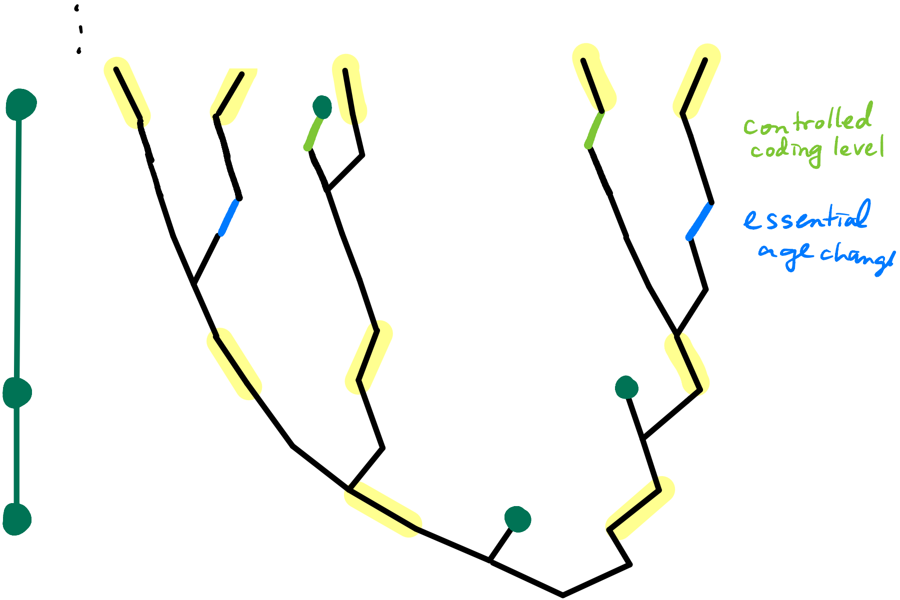
Theorem (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker, 2021)

We characterize the exact big Ramsey degrees of free amalgamation classes in finite languages with relations of arity at most two with finitely many forbidden finite irreducible substructures.

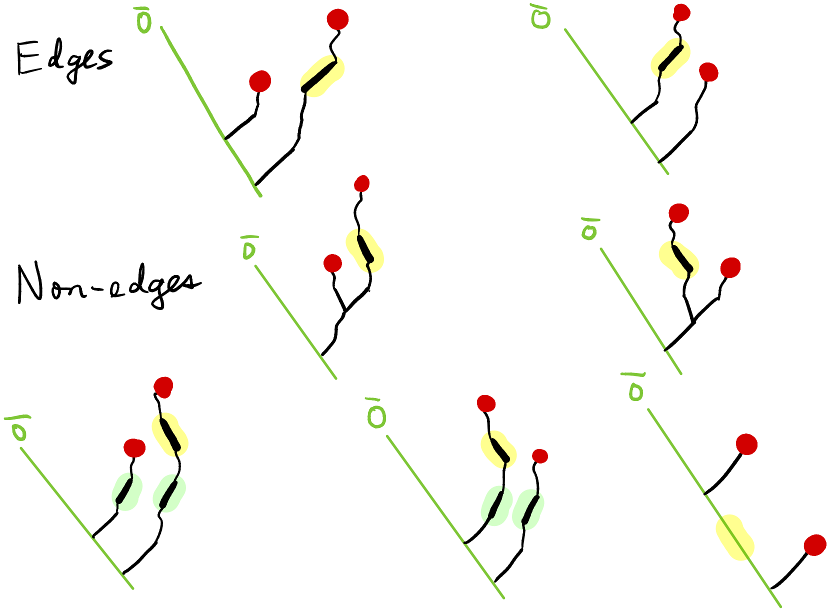
Keys of the characterization:

1. Diagonal antichains
2. Controlled splitting levels
3. Controlled coding levels
4. Controlled age-change levels (incremental changes in how much of a forbidden substructure is coded)
5. Controlled paths (only matter for non-trivial unary relations)

BRD in the Triangle-free Henson graph $\mathcal{H}_3$



BRD in the Triangle-free Henson graph $\mathcal{H}_3$



Infinite-dimensional Ramsey Theory on ω

Infinite-dimensional Ramsey Theory

A subset $\mathcal{X}$ of $[\omega]^\omega$ is **Ramsey** if each for $M \in [\omega]^\omega$, there is an $N \in [M]^\omega$ such that $[N]^\omega \subseteq \mathcal{X}$ or $[N]^\omega \cap \mathcal{X} = \emptyset$.

Infinite-dimensional Ramsey Theory

A subset $\mathcal{X}$ of $[\omega]^\omega$ is **Ramsey** if each for $M \in [\omega]^\omega$, there is an $N \in [M]^\omega$ such that $[N]^\omega \subseteq \mathcal{X}$ or $[N]^\omega \cap \mathcal{X} = \emptyset$.

Ramsey's Theorem (topological form). For any m and r, if $\mathcal{X} \subseteq [\omega]^\omega$ is a union of basic clopen sets of the form $[s, \omega]$ where $s \in [\omega]^m$, then $\mathcal{X}$ is Ramsey.

Infinite-dimensional Ramsey Theory

A subset $\mathcal{X}$ of $[\omega]^\omega$ is **Ramsey** if each for $M \in [\omega]^\omega$, there is an $N \in [M]^\omega$ such that $[N]^\omega \subseteq \mathcal{X}$ or $[N]^\omega \cap \mathcal{X} = \emptyset$.

AC $\implies \exists \mathcal{X} \subseteq [\omega]^\omega$ which is not Ramsey. (Erdős–Rado 1952)

Solution: restrict to 'definable' sets.

Infinite-dimensional Ramsey Theory

A subset $\mathcal{X}$ of $[\omega]^\omega$ is **Ramsey** if each for $M \in [\omega]^\omega$, there is an $N \in [M]^\omega$ such that $[N]^\omega \subseteq \mathcal{X}$ or $[N]^\omega \cap \mathcal{X} = \emptyset$.

Nash-Williams Thm. Clopen sets are Ramsey.

Galvin–Prikry Thm. Borel sets are Ramsey.

Silver Thm. Analytic sets are Ramsey.

Ellentuck Thm. A set is completely Ramsey iff it has the property of Baire in the Ellentuck topology.

Ellentuck Theorem

Ellentuck topology: refines the metric topology with basic open sets

$$[s, A] = \{B \in [\omega]^\omega : s \sqsubset B \subseteq A\}.$$

Theorem (Ellentuck)

A set $\mathcal{X} \subseteq [\omega]^\omega$ satisfies

$(*)$ $\quad \forall [s, A] \ \exists B \in [s, A]$ *such that* $[s, B] \subseteq \mathcal{X}$ *or* $[s, B] \cap \mathcal{X} = \emptyset$

iff $\mathcal{X}$ has the property of Baire with respect to the Ellentuck topology.

$(*)$ is called completely Ramsey by Galvin–Prikry and Ramsey by Todorcevic.

The Ellentuck space is the prototype for **topological Ramsey spaces**:

Ellentuck Theorem

Ellentuck topology: refines the metric topology with basic open sets

$$[s, A] = \{B \in [\omega]^\omega : s \sqsubset B \subseteq A\}.$$

Theorem (Ellentuck)

A set $\mathcal{X} \subseteq [\omega]^\omega$ satisfies

$(*)$ $\quad \forall [s, A] \ \exists B \in [s, A]$ *such that* $[s, B] \subseteq \mathcal{X}$ *or* $[s, B] \cap \mathcal{X} = \emptyset$

iff $\mathcal{X}$ has the property of Baire with respect to the Ellentuck topology.

$(*)$ is called completely Ramsey by Galvin–Prikry and Ramsey by Todorcevic.

The Ellentuck space is the prototype for **topological Ramsey spaces**: Points are infinite sequences, topology is induced by finite heads and infinite tails, and every subset with the property of Baire satisfies $(*)$.

KPT Question

Problem 11.2 in [KPT 2005]. Given a homogeneous structure $\mathbf{K}$, find the right notion of 'definable set' so that all definable subsets of $\binom{\mathbf{K}}{\mathbf{K}}$ are Ramsey.

We assume the universe of $\mathbf{K}$ is ω so that $\binom{\mathbf{K}}{\mathbf{K}}$ is a subspace of $[\omega]^\omega$.

Constraint: Big Ramsey degrees.

Must fix a Diary and work on subcopies (or embeddings) of it.

Infinite-Dimensional Ramsey Theory

Theorem (D. 2019)

The Rado graph has versions of the Galvin–Prikry Theorem.

Theorem (D. 2022)

Fraïssé structures satisfying SDAP$^+$ with finitely many relations of arity at most two have analogues of the Galvin–Prikry Theorem which directly recover big Ramsey degrees. Moreover, if $\mathbf{K}$ *has a certain amount of rigidity, we obtain analogues of Ellentuck's Theorem.*

Theorem (D., Zucker 2023)

Fix a finitely constrained binary free amalgamation class $\mathcal{K}$ *and let* $\mathbf{K} = Flim(\mathcal{K})$. *Then* $\mathbf{K}$ *has infinite-dimensional Ramsey theory which directly recovers exact big Ramsey degrees in (BCDHKVZ 2021).*

Abstract Ramsey Theorem

Theorem (Todorcevic)

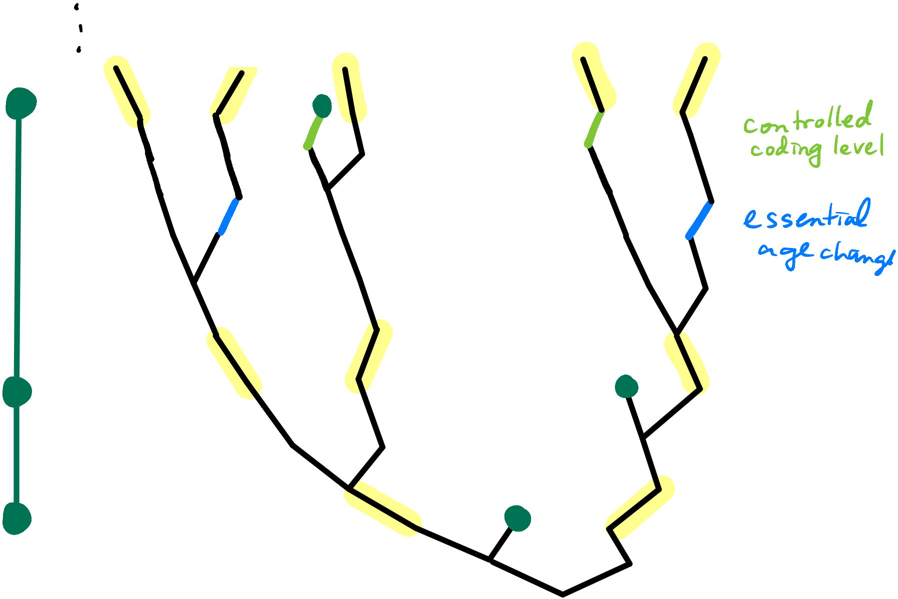
Suppose that $(\mathcal{R}, \mathcal{S}, \le, \le_{\mathcal{R}})$ with finite restriction maps satisfying axioms **A.1–A.4**, *and that $\mathcal{S}$ is closed. Then the field of $\mathcal{S}$-Ramsey subsets of $\mathcal{R}$ is closed under the Souslin operation and it coincides with the field of $\mathcal{S}$-Baire subsets of $\mathcal{R}$.*

When $\mathcal{R} = \mathcal{S}$, this theorem implies the Abstract Ellentuck Theorem.

Theorem (D., Zucker 2023)

The conclusion of the above theorem still holds when axiom **A.3(2)** *is replaced by the weaker existence of an* **A.3(2)**-*ideal.*

Triangle-free Henson Graph



Some Computability Theory Results on Big Ramsey Degrees

A List of Problems on the Reverse Mathematics of Ramsey Theory on the Rado Graph and on Infinite, Finitely Branching Trees
(D. arXiv:1808.10227)

Some results from *Milliken's tree theorem and its applications: a computability-theoretic perspective*, by Anglès d'Auriac, Cholak, Dzhafarov, Monin, Patey. AMS Memoirs (to appear)

Theorem (ACDMP)

1. *Big Ramsey degree theorems for the Rationals and the Rado graph are provable in ACA_0.*
2. *There is a computable coloring of pairs in the rationals all of whose subcopies with the minimal number of colors computes the halting set.*
3. *On the other hand, the Rado graph big Ramsey degrees has cone avoidance for pairs.*

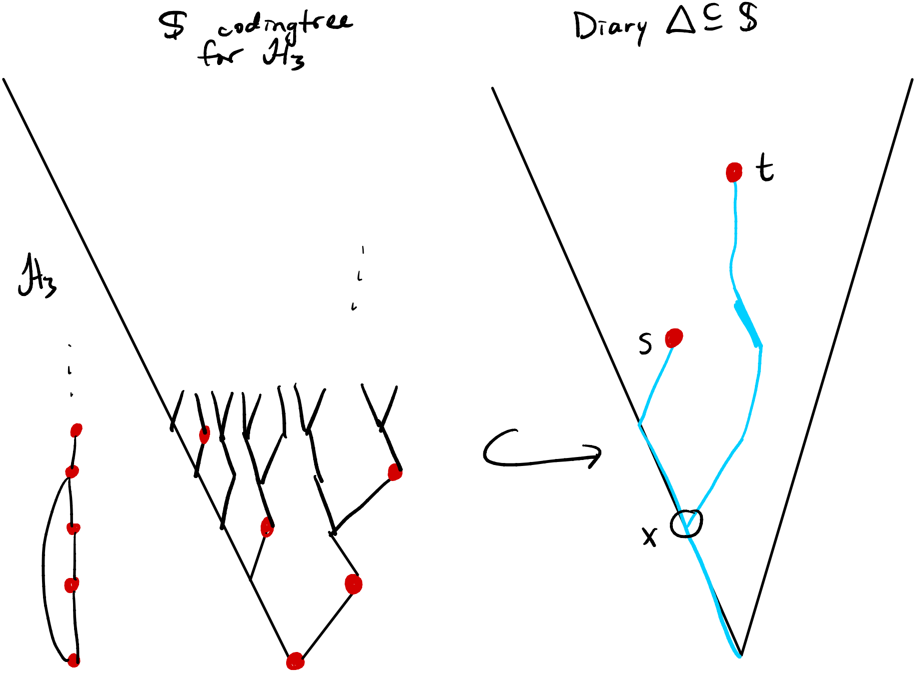
Theorem (Anglès d'Auriac, Liu, Mignoty, Patey)

(ACA_0^+) The triangle-free Henson graph admits finite big Ramsey degrees.

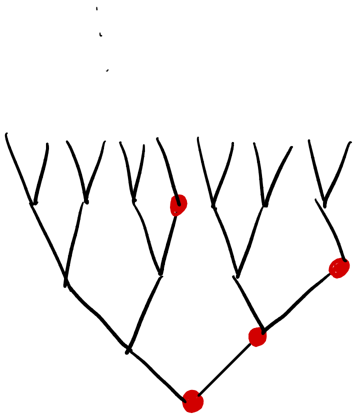
BRD in the Henson graphs

Theorem (Cholak, D., McCoy)

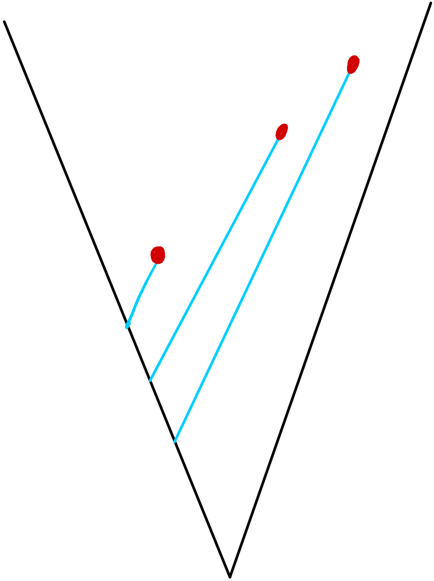
1. *There is a computable coloring of all pairs of non-edges of $\mathcal{H}_3$ so that any subcopy of $\mathcal{H}_3$ with the minimal number of colors computes $0'$.*
2. *More generally, for each $n \geq 3$, there is a finite n-clique-free graph $\mathbf{G}$ and a computable coloring of the copies of $\mathbf{G}$ in $\mathcal{H}_n$ so that any subcopy of $\mathcal{H}_n$ with the minimal number of colors computes $0^{(n-2)}$.*


$\mathbb{S}$ coding tree for $\mathcal{H}_3$
Diary $\Delta \subseteq \mathbb{S}$
$\mathcal{H}_3$
t
s
x

$\mathcal{H}_4$



Diary Δ



Some References

Milliken's tree theorem and its applications: a computability-theoretic perspective, by Anglès d'Auriac, Cholak, Dzhafarov, Monin, Patey. AMS Memoirs (to appear). arXiv:2007.09739

Dobrinen, *Ramsey theory of homogeneous structures: current trends and open problems*. Proceedings of the 2022 International Congress of Mathematicians (to appear). arXiv:2110.00655

Todorcevic, *Introduction to Ramsey Spaces*, Princeton Univ. Press, 2010.

Harrington's forcing proof of Halpern–Läuchli appears in its original form in

- Dobrinen, *Forcing in Ramsey theory*, RIMS Kokyuroku (2017) arXiv:1704.03898
- Dobrinen, Section 3.4 of *The Ramsey theory of Henson graphs*, JML 2023.

and in a stronger form with a harder proof in

- Farah and Todorcevic, *Some applications of the method of forcing*, Yenisei Series, 1995.

Thank You!