Arrow's theorem and the reverse mathematics of social choice theory

Benedict Eastaugh

University of Warwick

Computability and Combinatorics 2023

University of Connecticut, 19 May 2023

1. Social decision-making and Arrow's theorem

Majority rule and the Condorcet paradox

Majority rule: given alternatives x and y, if a majority of voters prefer x to y then the society should prefer x to y.

Majority rule and the Condorcet paradox

Majority rule: given alternatives x and y, if a majority of voters prefer x to y then the society should prefer x to y.

1. Suppose that voter A prefers x to y and y to z

Majority rule and the Condorcet paradox

Majority rule: given alternatives x and y, if a majority of voters prefer x to y then the society should prefer x to y.

1. Suppose that voter A prefers x to y and y to z,
   voter B prefers y to z and z to x

Majority rule and the Condorcet paradox

Majority rule: given alternatives x and y, if a majority of voters prefer x to y then the society should prefer x to y.

1. Suppose that voter A prefers x to y and y to z,
   voter B prefers y to z and z to x, and
   voter C prefers z to x and x to y.

Majority rule and the Condorcet paradox

Majority rule: given alternatives x and y, if a majority of voters prefer x to y then the society should prefer x to y.

1. Suppose that voter A prefers x to y and y to z,
   voter B prefers y to z and z to x, and
   voter C prefers z to x and x to y.
2. Then a majority prefer x to y (A and C)

Majority rule and the Condorcet paradox

Majority rule: given alternatives x and y, if a majority of voters prefer x to y then the society should prefer x to y.

1. Suppose that voter A prefers x to y and y to z,
   voter B prefers y to z and z to x, and
   voter C prefers z to x and x to y.
2. Then a majority prefer x to y (A and C),
   y to z (A and B)

Majority rule and the Condorcet paradox

Majority rule: given alternatives x and y, if a majority of voters prefer x to y then the society should prefer x to y.

1. Suppose that voter A prefers x to y and y to z,
   voter B prefers y to z and z to x, and
   voter C prefers z to x and x to y.
2. Then a majority prefer x to y (A and C),
   y to z (A and B), and
   z to x (B and C).

Majority rule and the Condorcet paradox

Majority rule: given alternatives x and y, if a majority of voters prefer x to y then the society should prefer x to y.

1. Suppose that voter A prefers x to y and y to z,
   voter B prefers y to z and z to x, and
   voter C prefers z to x and x to y.
2. Then a majority prefer x to y (A and C),
   y to z (A and B), and
   z to x (B and C).
3. While alternative should be chosen?

Majority rule and the Condorcet paradox

Majority rule: given alternatives x and y, if a majority of voters prefer x to y then the society should prefer x to y.

1. Suppose that voter A prefers x to y and y to z,
   voter B prefers y to z and z to x, and
   voter C prefers z to x and x to y.
2. Then a majority prefer x to y (A and C),
   y to z (A and B), and
   z to x (B and C).
3. While alternative should be chosen?
   - x is preferred to y, and y to z, so x should be preferable to z.

Majority rule and the Condorcet paradox

Majority rule: given alternatives x and y, if a majority of voters prefer x to y then the society should prefer x to y.

1. Suppose that voter A prefers x to y and y to z,
   voter B prefers y to z and z to x, and
   voter C prefers z to x and x to y.
2. Then a majority prefer x to y (A and C),
   y to z (A and B), and
   z to x (B and C).
3. While alternative should be chosen?
   - x is preferred to y, and y to z, so x should be preferable to z.
   - But in fact z is preferred to x.

Majority rule and the Condorcet paradox

Majority rule: given alternatives x and y, if a majority of voters prefer x to y then the society should prefer x to y.

1. Suppose that voter A prefers x to y and y to z,
   voter B prefers y to z and z to x, and
   voter C prefers z to x and x to y.
2. Then a majority prefer x to y (A and C),
   y to z (A and B), and
   z to x (B and C).
3. While alternative should be chosen?
   - x is preferred to y, and y to z, so x should be preferable to z.
   - But in fact z is preferred to x.

This is the **Condorcet paradox**.

Diagnosing the paradox

The preferences of the voters A, B, and C are transitive.

Diagnosing the paradox

The preferences of the voters A, B, and C are transitive. Even if voters are *individually* rational, majority rule can lead to *socially* irrational outcomes.

Diagnosing the paradox

The preferences of the voters A, B, and C are transitive. Even if voters are *individually* rational, majority rule can lead to *socially* irrational outcomes.

This suggests that we need to impose conditions on our aggregation functions to rule out such scenarios.

Diagnosing the paradox

The preferences of the voters A, B, and C are transitive. Even if voters are *individually* rational, majority rule can lead to *socially* irrational outcomes.

This suggests that we need to impose conditions on our aggregation functions to rule out such scenarios. With respect to the orderings they produce,

Diagnosing the paradox

The preferences of the voters A, B, and C are transitive. Even if voters are *individually* rational, majority rule can lead to *socially* irrational outcomes.

This suggests that we need to impose conditions on our aggregation functions to rule out such scenarios. With respect to the orderings they produce,

i. Social preferences should be transitive

Diagnosing the paradox

The preferences of the voters A, B, and C are transitive. Even if voters are *individually* rational, majority rule can lead to *socially* irrational outcomes.

This suggests that we need to impose conditions on our aggregation functions to rule out such scenarios. With respect to the orderings they produce,

i. Social preferences should be transitive;

ii. Any two alternatives should be comparable

Diagnosing the paradox

The preferences of the voters A, B, and C are transitive. Even if voters are *individually* rational, majority rule can lead to *socially* irrational outcomes.

This suggests that we need to impose conditions on our aggregation functions to rule out such scenarios. With respect to the orderings they produce,

i. Social preferences should be transitive;

ii. Any two alternatives should be comparable; and

iii. Ties should be permitted, to express indifference.

Diagnosing the paradox

The preferences of the voters A, B, and C are transitive. Even if voters are *individually* rational, majority rule can lead to *socially* irrational outcomes.

This suggests that we need to impose conditions on our aggregation functions to rule out such scenarios. With respect to the orderings they produce,

i. Social preferences should be transitive;
ii. Any two alternatives should be comparable; and
iii. Ties should be permitted, to express indifference.

This determines a notion of order known as a **weak order**.

Diagnosing the paradox

The preferences of the voters A, B, and C are transitive. Even if voters are *individually* rational, majority rule can lead to *socially* irrational outcomes.

This suggests that we need to impose conditions on our aggregation functions to rule out such scenarios. With respect to the orderings they produce,

i. Social preferences should be transitive;
ii. Any two alternatives should be comparable; and
iii. Ties should be permitted, to express indifference.

This determines a notion of order known as a **weak order**.

One can also work exclusively with **linear orders**.

Diagnosing the paradox

The preferences of the voters A, B, and C are transitive. Even if voters are *individually* rational, majority rule can lead to *socially* irrational outcomes.

This suggests that we need to impose conditions on our aggregation functions to rule out such scenarios. With respect to the orderings they produce,

i. Social preferences should be transitive;
ii. Any two alternatives should be comparable; and
iii. Ties should be permitted, to express indifference.

This determines a notion of order known as a **weak order**.

One can also work exclusively with **linear orders**. This makes some of the results easier, but also less interesting.

Weak orders

Definition

Let X be a set of *alternatives*. If $R \subseteq X \times X$ is such that

1. R is transitive, and
2. R is strongly connected, i.e. $(x, y) \in R \vee (y, x) \in R$ for all $x, y \in X$,

then we say R is a *weak order* on X.

Weak orders

Definition

Let X be a set of *alternatives*. If $R \subseteq X \times X$ is such that

1. R is transitive, and
2. R is strongly connected, i.e. $(x, y) \in R \vee (y, x) \in R$ for all $x, y \in X$,

then we say R is a *weak order* on X.

Some notation:

i. $x \lesssim_R y \equiv (x, y) \in R$

Weak orders

Definition

Let X be a set of *alternatives*. If $R \subseteq X \times X$ is such that

1. R is transitive, and
2. R is strongly connected, i.e. $(x, y) \in R \vee (y, x) \in R$ for all $x, y \in X$,

then we say R is a *weak order* on X.

Some notation:

i. $x \lesssim_R y \equiv (x, y) \in R$,

ii. $x <_R y \equiv (x, y) \in R \wedge (y, x) \notin R$

Weak orders

Definition

Let X be a set of *alternatives*. If $R \subseteq X \times X$ is such that

1. R is transitive, and
2. R is strongly connected, i.e. $(x, y) \in R \vee (y, x) \in R$ for all $x, y \in X$,

then we say R is a *weak order* on X.

Some notation:

i. $x \lesssim_R y \equiv (x, y) \in R$,

ii. $x <_R y \equiv (x, y) \in R \wedge (y, x) \notin R$,

iii. $x \sim_R y \equiv (x, y) \in R \wedge (y, x) \in R$

Weak orders

Definition

Let X be a set of *alternatives*. If $R \subseteq X \times X$ is such that

1. R is transitive, and
2. R is strongly connected, i.e. $(x, y) \in R \vee (y, x) \in R$ for all $x, y \in X$,

then we say R is a *weak order* on X.

Some notation:

i. $x \lesssim_R y \equiv (x, y) \in R$,

ii. $x <_R y \equiv (x, y) \in R \wedge (y, x) \notin R$,

iii. $x \sim_R y \equiv (x, y) \in R \wedge (y, x) \in R$,

iv. $W = \{R : R \text{ is a weak order on } X\}$.

Voters, profiles, coalitions

In general we can think of a society $\mathcal{S}$ in the Arrovian framework as having the following components

Voters, profiles, coalitions

In general we can think of a society $\mathcal{S}$ in the Arrovian framework as having the following components:

1. A nonempty set V of *voters*

Voters, profiles, coalitions

In general we can think of a society $\mathcal{S}$ in the Arrovian framework as having the following components:

1. A nonempty set V of *voters*,
2. A finite set X of *alternatives* and the set W of all weak orders on X

Voters, profiles, coalitions

In general we can think of a society $\mathcal{S}$ in the Arrovian framework as having the following components:

1. A nonempty set V of *voters*,
2. A finite set X of *alternatives* and the set W of all weak orders on X,
3. A set $\mathcal{A} \subseteq \mathcal{P}(V)$ of *coalitions* of voters

Voters, profiles, coalitions

In general we can think of a society $\mathcal{S}$ in the Arrovian framework as having the following components:

1. A nonempty set V of *voters*,
2. A finite set X of *alternatives* and the set W of all weak orders on X,
3. A set $\mathcal{A} \subseteq \mathcal{P}(V)$ of *coalitions* of voters, and
4. A set $\mathcal{F} \subseteq W^V$ of *profiles*, possible voting scenarios.

Voters, profiles, coalitions

In general we can think of a society $\mathcal{S}$ in the Arrovian framework as having the following components:

1. A nonempty set V of *voters*,
2. A finite set X of *alternatives* and the set W of all weak orders on X,
3. A set $\mathcal{A} \subseteq \mathcal{P}(V)$ of *coalitions* of voters, and
4. A set $\mathcal{F} \subseteq W^V$ of *profiles*, possible voting scenarios.

In Arrow's original setting, $\mathcal{A} = \mathcal{P}(V)$ and $\mathcal{F} = W^V$.

Voters, profiles, coalitions

In general we can think of a society $\mathcal{S}$ in the Arrovian framework as having the following components:

1. A nonempty set V of *voters*,
2. A finite set X of *alternatives* and the set W of all weak orders on X,
3. A set $\mathcal{A} \subseteq \mathcal{P}(V)$ of *coalitions* of voters, and
4. A set $\mathcal{F} \subseteq W^V$ of *profiles*, possible voting scenarios.

In Arrow's original setting, $\mathcal{A} = \mathcal{P}(V)$ and $\mathcal{F} = W^V$. This is known as the *unrestricted domain* condition.

More notation for profiles

To simplify the presentation of Arrow's conditions we need some more notation.

More notation for profiles

To simplify the presentation of Arrow's conditions we need some more notation.

i. Given a coalition $C \in \mathcal{A}$,

$$x \lesssim_{f[C]} y \equiv \forall v \in C(x \lesssim_{f(v)} y)$$

More notation for profiles

To simplify the presentation of Arrow's conditions we need some more notation.

i. Given a coalition $C \in \mathcal{A}$,

$$x \precsim_{f[C]} y \equiv \forall v \in C(x \precsim_{f(v)} y),$$

ii. Given $Y \subseteq X$ and $v \in V$, $f(v) = g(v)$ *on* Y if for all $x, y \in Y$,

$$x \precsim_{f(v)} y \leftrightarrow x \precsim_{g(v)} y$$

More notation for profiles

To simplify the presentation of Arrow's conditions we need some more notation.

i. Given a coalition $C \in \mathcal{A}$,

$$x \lesssim_{f[C]} y \equiv \forall v \in C(x \lesssim_{f(v)} y),$$

ii. Given $Y \subseteq X$ and $v \in V$, $f(v) = g(v)$ *on* Y if for all $x, y \in Y$,

$$x \lesssim_{f(v)} y \leftrightarrow x \lesssim_{g(v)} y,$$

iii. Given $Y \subseteq X$, $f = g$ *on* Y if $f(v) = g(v)$ on Y for all $v \in V$.

Social welfare functions

Given a society $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$, a *social welfare function* is a function $\sigma : \mathcal{F} \to W$.

Social welfare functions

Given a society $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$, a *social welfare function* is a function $\sigma : \mathcal{F} \to W$.

Arrow's conditions:

i. Unanimity

Social welfare functions

Given a society $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$, a *social welfare function* is a function $\sigma : \mathcal{F} \to W$.

Arrow's conditions:

i. Unanimity: If $x <_{f[V]} y$ then $x <_{\sigma(f)} y$.

Social welfare functions

Given a society $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$, a *social welfare function* is a function $\sigma : \mathcal{F} \to W$.

Arrow's conditions:

i. Unanimity: If $x <_{f[V]} y$ then $x <_{\sigma(f)} y$.
ii. Independence of irrelevant alternatives

Social welfare functions

Given a society $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$, a *social welfare function* is a function $\sigma : \mathcal{F} \to W$.

Arrow's conditions:

i. Unanimity: If $x <_{f[V]} y$ then $x <_{\sigma(f)} y$.

ii. Independence of irrelevant alternatives: For all $Y \subseteq X$ and all $f, g \in \mathcal{F}$, if $f = g$ on Y then $\sigma(f) = \sigma(g)$ on Y.

Social welfare functions

Given a society $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$, a *social welfare function* is a function $\sigma : \mathcal{F} \to W$.

Arrow's conditions:

i. Unanimity: If $x <_{f[V]} y$ then $x <_{\sigma(f)} y$.
ii. Independence of irrelevant alternatives: For all $Y \subseteq X$ and all $f, g \in \mathcal{F}$, if $f = g$ on Y then $\sigma(f) = \sigma(g)$ on Y.
iii. Non-dictatoriality

Social welfare functions

Given a society $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$, a *social welfare function* is a function $\sigma : \mathcal{F} \to W$.

Arrow's conditions:

i. Unanimity: If $x <_{f[V]} y$ then $x <_{\sigma(f)} y$.

ii. Independence of irrelevant alternatives: For all $Y \subseteq X$ and all $f, g \in \mathcal{F}$, if $f = g$ on Y then $\sigma(f) = \sigma(g)$ on Y.

iii. Non-dictatoriality: There exists no $d \in V$ such that for all $f \in \mathcal{F}$ and all $x, y \in X$, if $x <_{f(d)} y$ then $x <_{\sigma(f)} y$.

Arrow's impossibility theorem

Arrow's impossibility theorem (1950)

Suppose that $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a society which satisfies universal domain, V is finite, and $|X| \geq 3$.

Arrow's impossibility theorem

Arrow's impossibility theorem (1950)

Suppose that $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a society which satisfies universal domain, V is finite, and $|X| \geq 3$. Then there exists no social welfare function $\sigma : \mathcal{F} \to W$ which satisfies unanimity, independence, and non-dictatoriality.

Arrow's impossibility theorem

Arrow's impossibility theorem (1950)

Suppose that $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a society which satisfies universal domain, V is finite, and $|X| \geq 3$. Then there exists no social welfare function $\sigma : \mathcal{F} \to W$ which satisfies unanimity, independence, and non-dictatoriality.

> The main thrust of Arrow's theorem and all the associated literature is that there is an unresolvable tension between logicality and fairness. Independent choice requires concentration of power, in sharp conflict with democratic ideals.
>
> (Riker 1982, p. 136)

2. Ultrafilters and dictators

An escape from dictatoriality?

Infinite societies offer a kind of escape from Arrow's theorem.

An escape from dictatoriality?

Infinite societies offer a kind of escape from Arrow's theorem.

Fishburn's possibility theorem (1970)

Suppose that $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a society which satisfies universal domain and V is infinite.

An escape from dictatoriality?

Infinite societies offer a kind of escape from Arrow's theorem.

Fishburn's possibility theorem (1970)

Suppose that $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a society which satisfies universal domain and V is infinite. Then there exists a social welfare function $\sigma : \mathcal{F} \to W$ which satisfies unanimity, independence, and non-dictatoriality.

An escape from dictatoriality?

Infinite societies offer a kind of escape from Arrow's theorem.

Fishburn's possibility theorem (1970)

Suppose that $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a society which satisfies universal domain and V is infinite. Then there exists a social welfare function $\sigma : \mathcal{F} \to W$ which satisfies unanimity, independence, and non-dictatoriality.

However, Fishburn's proof uses (a fragment of) the axiom of choice, in the guise of a non-principal ultrafilter on $\mathcal{A}$.

An escape from dictatoriality?

Infinite societies offer a kind of escape from Arrow's theorem.

Fishburn's possibility theorem (1970)

Suppose that $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a society which satisfies universal domain and V is infinite. Then there exists a social welfare function $\sigma : \mathcal{F} \to W$ which satisfies unanimity, independence, and non-dictatoriality.

However, Fishburn's proof uses (a fragment of) the axiom of choice, in the guise of a non-principal ultrafilter on $\mathcal{A}$.

Fishburn's possibility theorem therefore sparked debate amongst social choice theorists concerning the use of non-constructive methods (see e.g. Litak 2018).

Decisive coalitions

In the course of his proof of the impossibility theorem, Arrow introduced the notion of a *decisive coalition*.

Decisive coalitions

In the course of his proof of the impossibility theorem, Arrow introduced the notion of a *decisive coalition*.

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a society.

Decisive coalitions

In the course of his proof of the impossibility theorem, Arrow introduced the notion of a *decisive coalition*.

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a society. Then

i. $C \in \mathcal{A}$ is σ-*decisive* if for all $f \in \mathcal{F}$ and all $x, y \in X$,

$$x <_{f[C]} y \rightarrow x <_{\sigma(f)} y.$$

Decisive coalitions

In the course of his proof of the impossibility theorem, Arrow introduced the notion of a *decisive coalition*.

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a society. Then

i. $C \in \mathcal{A}$ is σ-*decisive* if for all $f \in \mathcal{F}$ and all $x, y \in X$,

$$x <_{f[C]} y \rightarrow x <_{\sigma(f)} y.$$

ii. $d \in V$ is *dictatorial* if $\{d\}$ is σ-decisive.

The Kirman–Sondermann theorem

The notion of a decisive coalition forms the basis of an influential analysis of Arrow's theorem in terms of ultrafilters.

The Kirman–Sondermann theorem

The notion of a decisive coalition forms the basis of an influential analysis of Arrow's theorem in terms of ultrafilters.

Kirman–Sondermann theorem (1972)

Suppose that $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a society which satisfies universal domain, $|X| \geq 3$, and $\sigma : \mathcal{F} \to W$ is a social welfare function satisfying unanimity and independence.

The Kirman–Sondermann theorem

The notion of a decisive coalition forms the basis of an influential analysis of Arrow's theorem in terms of ultrafilters.

Kirman–Sondermann theorem (1972)

Suppose that $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a society which satisfies universal domain, $|X| \geq 3$, and $\sigma : \mathcal{F} \to W$ is a social welfare function satisfying unanimity and independence. Then there exists an ultrafilter $\mathcal{U}_\sigma$ on $\mathcal{A}$ such that

$$\mathcal{U}_\sigma = \{C \in \mathcal{A} : C \text{ is } \sigma\text{-decisive}\}$$

The Kirman–Sondermann theorem

The notion of a decisive coalition forms the basis of an influential analysis of Arrow's theorem in terms of ultrafilters.

Kirman–Sondermann theorem (1972)

Suppose that $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a society which satisfies universal domain, $|X| \geq 3$, and $\sigma : \mathcal{F} \to W$ is a social welfare function satisfying unanimity and independence. Then there exists an ultrafilter $\mathcal{U}_\sigma$ on $\mathcal{A}$ such that

$$\mathcal{U}_\sigma = \{C \in \mathcal{A} : C \text{ is } \sigma\text{-decisive}\},$$

and $\mathcal{U}_\sigma$ is principal if and only if σ is dictatorial.

The Kirman–Sondermann theorem

The notion of a decisive coalition forms the basis of an influential analysis of Arrow's theorem in terms of ultrafilters.

Kirman–Sondermann theorem (1972)

Suppose that $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a society which satisfies universal domain, $|X| \geq 3$, and $\sigma : \mathcal{F} \to W$ is a social welfare function satisfying unanimity and independence. Then there exists an ultrafilter $\mathcal{U}_\sigma$ on $\mathcal{A}$ such that

$$\mathcal{U}_\sigma = \{C \in \mathcal{A} : C \text{ is } \sigma\text{-decisive}\},$$

and $\mathcal{U}_\sigma$ is principal if and only if σ is dictatorial.

This is provable in ZF, and therefore Fishburn's possibility theorem is not provable in ZF, since it implies the existence of a non-principal ultrafilter on $\mathcal{P}(V)$ for any infinite set V.

3. Countable social choice theory

Challenges in formalising social choice theory

Two principal challenges when formalising social choice theory in $\mathcal{L}_2$

Challenges in formalising social choice theory

Two principal challenges when formalising social choice theory in $\mathcal{L}_2$:

1. **Cardinality.**

Challenges in formalising social choice theory

Two principal challenges when formalising social choice theory in $\mathcal{L}_2$:

1. **Cardinality.**
   - When $V \subseteq \mathbb{N}$ is infinite, the unrestricted domain condition that $\mathcal{A} = \mathcal{P}(V)$ and $\mathcal{F} = W^V$ means that
     $$|\mathcal{A}| = |\mathcal{F}| = 2^{\aleph_0}.$$

Challenges in formalising social choice theory

Two principal challenges when formalising social choice theory in $\mathcal{L}_2$:

1. **Cardinality.**
   - When $V \subseteq \mathbb{N}$ is infinite, the unrestricted domain condition that $\mathcal{A} = \mathcal{P}(V)$ and $\mathcal{F} = W^V$ means that
   $$|\mathcal{A}| = |\mathcal{F}| = 2^{\aleph_0}.$$
   - In such cases social welfare functions $\sigma : \mathcal{F} \to W$ and ultrafilters $\mathcal{U}_\sigma \subseteq \mathcal{A}$ are also uncountable objects.

Challenges in formalising social choice theory

Two principal challenges when formalising social choice theory in $\mathcal{L}_2$:

1. **Cardinality.**
   - When $V \subseteq \mathbb{N}$ is infinite, the unrestricted domain condition that $\mathcal{A} = \mathcal{P}(V)$ and $\mathcal{F} = W^V$ means that
   $$|\mathcal{A}| = |\mathcal{F}| = 2^{\aleph_0}.$$
   - In such cases social welfare functions $\sigma : \mathcal{F} \to W$ and ultrafilters $\mathcal{U}_\sigma \subseteq \mathcal{A}$ are also uncountable objects.
2. **Effectiveness.**

Challenges in formalising social choice theory

Two principal challenges when formalising social choice theory in $\mathcal{L}_2$:

1. **Cardinality.**
   - When $V \subseteq \mathbb{N}$ is infinite, the unrestricted domain condition that $\mathcal{A} = \mathcal{P}(V)$ and $\mathcal{F} = W^V$ means that
   $$|\mathcal{A}| = |\mathcal{F}| = 2^{\aleph_0}.$$
   - In such cases social welfare functions $\sigma : \mathcal{F} \to W$ and ultrafilters $\mathcal{U}_\sigma \subseteq \mathcal{A}$ are also uncountable objects.
2. **Effectiveness.**
   - Definitions need to be tractable in the base theory.

Approaches to ultrafilters in reverse mathematics

1. Use a higher-order framework like that of Kohlenbach (2002, 2005).

Approaches to ultrafilters in reverse mathematics

1. Use a higher-order framework like that of Kohlenbach (2002, 2005).

2. Follow Towsner (2014) and extend $\mathcal{L}_2$ with third-order predicates Σ and $\mathfrak{U}$ for the social welfare function and corresponding ultrafilter.

Approaches to ultrafilters in reverse mathematics

1. Use a higher-order framework like that of Kohlenbach (2002, 2005).

2. Follow Towsner (2014) and extend $\mathcal{L}_2$ with third-order predicates Σ and $\mathfrak{U}$ for the social welfare function and corresponding ultrafilter.

3. Restrict our attention to *countable societies*: ones in which V, $\mathcal{A}$, and $\mathcal{F}$ are all countable.

Approaches to ultrafilters in reverse mathematics

1. Use a higher-order framework like that of Kohlenbach (2002, 2005).
2. Follow Towsner (2014) and extend $\mathcal{L}_2$ with third-order predicates Σ and $\mathfrak{U}$ for the social welfare function and corresponding ultrafilter.
3. Restrict our attention to *countable societies*: ones in which V, $\mathcal{A}$, and $\mathcal{F}$ are all countable.
   - This path is the one we will follow.

Voters V, alternatives X, profiles $\mathcal{F}$

- The set of voters will be a (finite or infinite) set $V \subseteq \mathbb{N}$.

Voters V, alternatives X, profiles $\mathcal{F}$

- The set of voters will be a (finite or infinite) set $V \subseteq \mathbb{N}$.
- The alternatives will be a finite set $X \subseteq \mathbb{N}$ such that $|X| \geq 3$, with W the set of all weak orders on X.

Voters V, alternatives X, profiles $\mathcal{F}$

- The set of voters will be a (finite or infinite) set $V \subseteq \mathbb{N}$.
- The alternatives will be a finite set $X \subseteq \mathbb{N}$ such that $|X| \geq 3$, with W the set of all weak orders on X.
- To address the **cardinality** issue we weaken the unrestricted domain condition.

Voters V, alternatives X, profiles $\mathcal{F}$

- The set of voters will be a (finite or infinite) set $V \subseteq \mathbb{N}$.
- The alternatives will be a finite set $X \subseteq \mathbb{N}$ such that $|X| \geq 3$, with W the set of all weak orders on X.
- To address the **cardinality** issue we weaken the unrestricted domain condition.

  We consider countable sets of profiles $\mathcal{F}$ coded in the usual way as countable sequences $\langle f_i : i \in \mathbb{N} \rangle$ where $f_i : V \to W$ for all $i \in \mathbb{N}$.

Voters V, alternatives X, profiles $\mathcal{F}$

- The set of voters will be a (finite or infinite) set $V \subseteq \mathbb{N}$.
- The alternatives will be a finite set $X \subseteq \mathbb{N}$ such that $|X| \geq 3$, with W the set of all weak orders on X.
- To address the **cardinality** issue we weaken the unrestricted domain condition.

  We consider countable sets of profiles $\mathcal{F}$ coded in the usual way as countable sequences $\langle f_i : i \in \mathbb{N} \rangle$ where $f_i : V \to W$ for all $i \in \mathbb{N}$.
  - To save ourselves from too many subscripts of subscripts of subscripts, we write

    $$x \precsim_{i(v)} y$$

    to mean that $x \precsim_R y$ where $R = f_i(v)$.

Countable algebras of coalitions

We also restrict the set of coalitions in any given society to a countable algebra.

Countable algebras of coalitions

We also restrict the set of coalitions in any given society to a countable algebra.

Definition (countable algebras of sets)

An *atomic countable algebra of sets* over V is a sequence $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ of subsets of V which contains V and all singletons $\{v\}$ for $v \in V$, and is closed under unions, complements, and intersections.

Countable algebras of coalitions

We also restrict the set of coalitions in any given society to a countable algebra.

Definition (countable algebras of sets)

An *atomic countable algebra of sets* over V is a sequence $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ of subsets of V which contains V and all singletons $\{v\}$ for $v \in V$, and is closed under unions, complements, and intersections.

- This kind of move was anticipated by Armstrong (1980), who generalised Kirman and Sondermann's results to algebras of coalitions $\mathcal{A} \subseteq \mathcal{P}(V)$.

Countable algebras of coalitions

We also restrict the set of coalitions in any given society to a countable algebra.

Definition (countable algebras of sets)

An *atomic countable algebra of sets* over V is a sequence $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ of subsets of V which contains V and all singletons $\{v\}$ for $v \in V$, and is closed under unions, complements, and intersections.

- This kind of move was anticipated by Armstrong (1980), who generalised Kirman and Sondermann's results to algebras of coalitions $\mathcal{A} \subseteq \mathcal{P}(V)$.

It will sometimes be necessary to compute unions, intersections, and complementation in a uniform way

Countable algebras of coalitions

We also restrict the set of coalitions in any given society to a countable algebra.

Definition (countable algebras of sets)

An *atomic countable algebra of sets* over V is a sequence $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ of subsets of V which contains V and all singletons $\{v\}$ for $v \in V$, and is closed under unions, complements, and intersections.

- This kind of move was anticipated by Armstrong (1980), who generalised Kirman and Sondermann's results to algebras of coalitions $\mathcal{A} \subseteq \mathcal{P}(V)$.

It will sometimes be necessary to compute unions, intersections, and complementation in a uniform way, so assume that we work only with algebras that have been computably reordered to facilitate this.

Measurable profiles

Having moved to countable algebras, we face a problem of **richness**

Measurable profiles

Having moved to countable algebras, we face a problem of **richness**: we need to ensure that $\mathcal{A}$ and $\mathcal{F}$ are *sufficiently rich* to allow the proofs to go through.

Measurable profiles

Having moved to countable algebras, we face a problem of **richness**: we need to ensure that $\mathcal{A}$ and $\mathcal{F}$ are *sufficiently rich* to allow the proofs to go through.

- Armstrong (1980) solves this problem by letting $\mathcal{F}$ consist of all $\mathcal{A}$-measurable profiles, i.e. every $f \in W^V$ such that for all $x, y \in X$,

$$\{v : x \precsim_{f(v)} y\} \in \mathcal{A}.$$

Measurable profiles

Having moved to countable algebras, we face a problem of **richness**: we need to ensure that $\mathcal{A}$ and $\mathcal{F}$ are *sufficiently rich* to allow the proofs to go through.

- Armstrong (1980) solves this problem by letting $\mathcal{F}$ consist of all $\mathcal{A}$-measurable profiles, i.e. every $f \in W^V$ such that for all $x, y \in X$,

$$\{v : x \precsim_{f(v)} y\} \in \mathcal{A}.$$

- But taking this approach provides us with a problem of **effectivity** since being $\mathcal{A}$-measurable is a Σ^0_2 predicate: $\exists k \forall v (x \precsim_{f_i(v)} y \leftrightarrow v \in A_k)$.

Measurable profiles

Having moved to countable algebras, we face a problem of **richness**: we need to ensure that $\mathcal{A}$ and $\mathcal{F}$ are *sufficiently rich* to allow the proofs to go through.

- Armstrong (1980) solves this problem by letting $\mathcal{F}$ consist of all $\mathcal{A}$-measurable profiles, i.e. every $f \in W^V$ such that for all $x, y \in X$,

$$\{v : x \precsim_{f(v)} y\} \in \mathcal{A}.$$

- But taking this approach provides us with a problem of **effectivity** since being $\mathcal{A}$-measurable is a Σ^0_2 predicate: $\exists k \forall v (x \precsim_{f_i(v)} y \leftrightarrow v \in A_k)$.
- We therefore require that for every $i \in \mathbb{N}$, f_i is $\mathcal{A}$-measurable via a uniformising function built into the definition of a society.

Uniform measurability

Definition (uniform $\mathcal{A}$-measurability)

Suppose $V \subseteq \mathbb{N}$ is nonempty and $X \subseteq \mathbb{N}$ is finite with $|X| \geq 3$, and that $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ is a countable algebra of sets over V and $\mathcal{F} = \langle f_i : i \in \mathbb{N} \rangle$ is a countable sequence of profiles.

Uniform measurability

Definition (uniform $\mathcal{A}$-measurability)

Suppose $V \subseteq \mathbb{N}$ is nonempty and $X \subseteq \mathbb{N}$ is finite with $|X| \geq 3$, and that $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ is a countable algebra of sets over V and $\mathcal{F} = \langle f_i : i \in \mathbb{N} \rangle$ is a countable sequence of profiles.

If there exists $\theta : \mathbb{N} \times X \times X \to \mathbb{N}$ such that for all $m \in \mathbb{N}$, $x, y \in X$,

$$\{v : x \lesssim_{m(v)} y\} = A_{\theta(m,x,y)},$$

then we say $\mathcal{F}$ is *uniformly* $\mathcal{A}$*-measurable* via θ.

Quasi-partition embedding: sketch

Another side of the problem of richness is that $\mathcal{F}$ needs to be sufficiently combinatorially rich with respect to $\mathcal{A}$.

Quasi-partition embedding: sketch

Another side of the problem of richness is that $\mathcal{F}$ needs to be sufficiently combinatorially rich with respect to $\mathcal{A}$.

- In particular, $\mathcal{F}$ needs to contain profiles encoding membership not just in individual elements of $\mathcal{A}$, but in finite partitions of V.

Quasi-partition embedding: sketch

Another side of the problem of richness is that $\mathcal{F}$ needs to be sufficiently combinatorially rich with respect to $\mathcal{A}$.

- In particular, $\mathcal{F}$ needs to contain profiles encoding membership not just in individual elements of $\mathcal{A}$, but in finite partitions of V.
- However, checking whether a finite sequence of indexes in $\mathcal{A}$ is a partition of V is a non-computable problem. So we can't demand that all partitions (even of bounded length) are coded into $\mathcal{F}$.

Quasi-partition embedding: sketch

Another side of the problem of richness is that $\mathcal{F}$ needs to be sufficiently combinatorially rich with respect to $\mathcal{A}$.

- In particular, $\mathcal{F}$ needs to contain profiles encoding membership not just in individual elements of $\mathcal{A}$, but in finite partitions of V.
- However, checking whether a finite sequence of indexes in $\mathcal{A}$ is a partition of V is a non-computable problem. So we can't demand that all partitions (even of bounded length) are coded into $\mathcal{F}$.
- This leads us to the idea of a *quasi-partition*, a finite sequence $s \in \mathrm{Seq}$ of indexes in $\mathcal{A}$ in which overlaps are allowed.

Quasi-partition embedding: sketch

Another side of the problem of richness is that $\mathcal{F}$ needs to be sufficiently combinatorially rich with respect to $\mathcal{A}$.

- In particular, $\mathcal{F}$ needs to contain profiles encoding membership not just in individual elements of $\mathcal{A}$, but in finite partitions of V.
- However, checking whether a finite sequence of indexes in $\mathcal{A}$ is a partition of V is a non-computable problem. So we can't demand that all partitions (even of bounded length) are coded into $\mathcal{F}$.
- This leads us to the idea of a *quasi-partition*, a finite sequence $s \in \mathrm{Seq}$ of indexes in $\mathcal{A}$ in which overlaps are allowed.
- Say $\mathcal{A}$ is *quasi-partition embedded* into $\mathcal{F}$ if there is a map e such that for any quasi-partition $s \in \mathrm{Seq}$ and permutation p of W with $|s| \leq |p|$, $f_{e(p,s)}$ is a profile which sends the elements that appear in (only) a given set in the quasi-partition to a unique weak order.

Quasi-partition embedding: definition

Definition (quasi-partition embedding)

Suppose $V \subseteq \mathbb{N}$ is nonempty and $X \subseteq \mathbb{N}$ is finite with $|X| \geq 3$, and that $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ is a countable algebra of sets over V and $\mathcal{F} = \langle f_i : i \in \mathbb{N} \rangle$ is a countable sequence of profiles.

Quasi-partition embedding: definition

Definition (quasi-partition embedding)

Suppose $V \subseteq \mathbb{N}$ is nonempty and $X \subseteq \mathbb{N}$ is finite with $|X| \geq 3$, and that $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ is a countable algebra of sets over V and $\mathcal{F} = \langle f_i : i \in \mathbb{N} \rangle$ is a countable sequence of profiles.

1. A *permutation* of the set W of (codes for) weak orders over X is a finite sequence $p \in \mathrm{Seq}$ such that for all $R \in W$ there exists a unique $i < |p|$ such that $p(i) = R$. If p is a permutation of W we write $p \in \mathrm{Perm}(W)$.

Quasi-partition embedding: definition

Definition (quasi-partition embedding)

Suppose $V \subseteq \mathbb{N}$ is nonempty and $X \subseteq \mathbb{N}$ is finite with $|X| \geq 3$, and that $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ is a countable algebra of sets over V and $\mathcal{F} = \langle f_i : i \in \mathbb{N} \rangle$ is a countable sequence of profiles.

1. A *permutation* of the set W of (codes for) weak orders over X is a finite sequence $p \in \mathrm{Seq}$ such that for all $R \in W$ there exists a unique $i < |p|$ such that $p(i) = R$. If p is a permutation of W we write $p \in \mathrm{Perm}(W)$.
2. A *quasi-partition* is a finite sequence s such that $|s| \geq 1$. If s is a quasi-partition with $|s| = k$, we write $s \in \mathrm{QPart}(k)$.

Quasi-partition embedding: definition

Definition (quasi-partition embedding)

Suppose $V \subseteq \mathbb{N}$ is nonempty and $X \subseteq \mathbb{N}$ is finite with $|X| \geq 3$, and that $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ is a countable algebra of sets over V and $\mathcal{F} = \langle f_i : i \in \mathbb{N} \rangle$ is a countable sequence of profiles.

1. A *permutation* of the set W of (codes for) weak orders over X is a finite sequence $p \in \mathrm{Seq}$ such that for all $R \in W$ there exists a unique $i < |p|$ such that $p(i) = R$. If p is a permutation of W we write $p \in \mathrm{Perm}(W)$.
2. A *quasi-partition* is a finite sequence s such that $|s| \geq 1$. If s is a quasi-partition with $|s| = k$, we write $s \in \mathrm{QPart}(k)$.
3. $\mathcal{A}$ is *quasi-partition embedded* into $\mathcal{F}$ via e if there exists a function $e : \mathrm{Perm}(W) \times \mathrm{QPart}(|W|) \to \mathbb{N}$ such that for all $v \in V$,

$$f_{e(p,s)}(v) = \begin{cases} p(i) & \text{if } (\exists! i < |s| - 1)(v \in A_{s(i)}), \\ p(|s| - 1) & \text{otherwise.} \end{cases}$$

Countable societies in $\mathcal{L}_2$

Combining these definitions we finally reach the definition of a society in $\mathcal{L}_2$.

Countable societies in $\mathcal{L}_2$

Combining these definitions we finally reach the definition of a society in $\mathcal{L}_2$.

Definition

$\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a *countable society* if $V \subseteq \mathbb{N}$ is nonempty, $X \subseteq \mathbb{N}$ is finite with $|X| \geq 3$, $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ is an atomic countable algebra, and $\mathcal{F} = \langle f_i : i \in \mathbb{N} \rangle$ is a countable sequence of profiles such that

1. $\mathcal{F}$ is uniformly $\mathcal{A}$-measurable, and
2. $\mathcal{A}$ is quasi-partition embedded into $\mathcal{F}$.

A countable society is *finite* if V is finite, and *infinite* otherwise.

Social welfare functions in $\mathcal{L}_2$

- From this point on we consider only social welfare functions satisfying unanimity (1) and independence (2).

Social welfare functions in $\mathcal{L}_2$

- From this point on we consider only social welfare functions satisfying unanimity (1) and independence (2).

Definition

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society. A function $\sigma : \mathbb{N} \to W$ is a *social welfare function* for $\mathcal{S}$ if

Social welfare functions in $\mathcal{L}_2$

- From this point on we consider only social welfare functions satisfying unanimity (1) and independence (2).

Definition

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society. A function $\sigma : \mathbb{N} \to W$ is a *social welfare function* for $\mathcal{S}$ if:

1. For all $i \in \mathbb{N}$ and all $x, y \in X$, if $x <_{i[V]} y$ then $x <_{\sigma(i)} y$.

Social welfare functions in $\mathcal{L}_2$

- From this point on we consider only social welfare functions satisfying unanimity (1) and independence (2).

Definition

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society. A function $\sigma : \mathbb{N} \to W$ is a *social welfare function* for $\mathcal{S}$ if:

1. For all $i \in \mathbb{N}$ and all $x, y \in X$, if $x <_{i[V]} y$ then $x <_{\sigma(i)} y$.
2. For all $i, j \in \mathbb{N}$ and all $Y \subseteq X$, if $f_i = f_j$ on Y then $\sigma(i) = \sigma(j)$ on Y.

Social welfare functions in $\mathcal{L}_2$

- From this point on we consider only social welfare functions satisfying unanimity (1) and independence (2).

Definition

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society. A function $\sigma : \mathbb{N} \to W$ is a *social welfare function* for $\mathcal{S}$ if:

1. For all $i \in \mathbb{N}$ and all $x, y \in X$, if $x <_{i[V]} y$ then $x <_{\sigma(i)} y$.
2. For all $i, j \in \mathbb{N}$ and all $Y \subseteq X$, if $f_i = f_j$ on Y then $\sigma(i) = \sigma(j)$ on Y.

If the following additional condition is satisfied then we say σ is *non-dictatorial*:

3. For all $v \in V$ there exists $i \in \mathbb{N}$ and $x, y \in X$ such that $x <_{i(v)} y$ and $y \lesssim_{\sigma(i)} x$.

Arrow's theorem and related statements in $\mathcal{L}_2$

Definition

The *Kirman–Sondermann theorem for countable societies* (KS) is the statement that for all countable societies $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ and all social welfare functions σ for $\mathcal{S}$, the set

$$\mathcal{U}_\sigma = \{i : A_i \text{ is } \sigma\text{-decisive}\}$$

of σ-decisive coalitions exists and forms an ultrafilter on $\mathcal{A}$, and $\mathcal{U}_\sigma$ is principal if and only if σ is dictatorial.

Arrow's theorem and related statements in $\mathcal{L}_2$

Definition

The *Kirman–Sondermann theorem for countable societies* (KS) is the statement that for all countable societies $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ and all social welfare functions σ for $\mathcal{S}$, the set

$$\mathcal{U}_\sigma = \{i : A_i \text{ is } \sigma\text{-decisive}\}$$

of σ-decisive coalitions exists and forms an ultrafilter on $\mathcal{A}$, and $\mathcal{U}_\sigma$ is principal if and only if σ is dictatorial.

Arrow's theorem is the statement that for all finite societies $\mathcal{S}$, if σ is a social welfare function for $\mathcal{S}$ then σ is dictatorial.

Arrow's theorem and related statements in $\mathcal{L}_2$

Definition

The *Kirman–Sondermann theorem for countable societies* (KS) is the statement that for all countable societies $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ and all social welfare functions σ for $\mathcal{S}$, the set

$$\mathcal{U}_\sigma = \{i : A_i \text{ is } \sigma\text{-decisive}\}$$

of σ-decisive coalitions exists and forms an ultrafilter on $\mathcal{A}$, and $\mathcal{U}_\sigma$ is principal if and only if σ is dictatorial.

Arrow's theorem is the statement that for all finite societies $\mathcal{S}$, if σ is a social welfare function for $\mathcal{S}$ then σ is dictatorial.

Fishburn's possibility theorem for countable societies (FPT) is the statement that for all countably infinite societies $\mathcal{S}$ there exists a non-dictatorial social welfare function σ for $\mathcal{S}$.

4. Proving Arrow's theorem

Almost σ-decisive coalitions

We say that A_n is *almost σ-decisive for x, y at i* if

$$x <_{i[A_n]} y \wedge y <_{i[A_n^c]} x \wedge x <_{\sigma(i)} y,$$

and A_n is *almost σ-decisive* if for all $i \in \mathbb{N}$ and $x, y \in X$,

$$(x <_{i[A_n]} y \wedge y <_{i[A_n^c]} x) \to x <_{\sigma(i)} y.$$

Almost σ-decisive coalitions

We say that A_n is *almost σ-decisive for x, y at i* if

$$x <_{i[A_n]} y \wedge y <_{i[A_n^c]} x \wedge x <_{\sigma(i)} y,$$

and A_n is *almost σ-decisive* if for all $i \in \mathbb{N}$ and $x, y \in X$,

$$(x <_{i[A_n]} y \wedge y <_{i[A_n^c]} x) \rightarrow x <_{\sigma(i)} y.$$

Definability lemma for σ-decisiveness

The following is provable in RCA_0. Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society. Then there exists $g : \mathbb{N} \to \mathbb{N}$ and $x, y \in X$ such that the following are equivalent for all $n \in \mathbb{N}$.

Almost σ-decisive coalitions

We say that A_n is *almost σ-decisive for x, y at i* if

$$x <_{i[A_n]} y \wedge y <_{i[A_n^c]} x \wedge x <_{\sigma(i)} y,$$

and A_n is *almost σ-decisive* if for all $i \in \mathbb{N}$ and $x, y \in X$,

$$(x <_{i[A_n]} y \wedge y <_{i[A_n^c]} x) \rightarrow x <_{\sigma(i)} y.$$

Definability lemma for σ-decisiveness

The following is provable in RCA_0. Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society. Then there exists $g : \mathbb{N} \rightarrow \mathbb{N}$ and $x, y \in X$ such that the following are equivalent for all $n \in \mathbb{N}$.

1. A_n is σ-decisive.

Almost σ-decisive coalitions

We say that A_n is *almost σ-decisive for x, y at i* if

$$x <_{i[A_n]} y \wedge y <_{i[A_n^c]} x \wedge x <_{\sigma(i)} y,$$

and A_n is *almost σ-decisive* if for all $i \in \mathbb{N}$ and $x, y \in X$,

$$(x <_{i[A_n]} y \wedge y <_{i[A_n^c]} x) \rightarrow x <_{\sigma(i)} y.$$

Definability lemma for σ-decisiveness

The following is provable in RCA_0. Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society. Then there exists $g : \mathbb{N} \rightarrow \mathbb{N}$ and $x, y \in X$ such that the following are equivalent for all $n \in \mathbb{N}$.

1. A_n is σ-decisive.
2. A_n is almost σ-decisive.

Almost σ-decisive coalitions

We say that A_n is *almost σ-decisive for x, y at i* if

$$x <_{i[A_n]} y \wedge y <_{i[A_n^c]} x \wedge x <_{\sigma(i)} y,$$

and A_n is *almost σ-decisive* if for all $i \in \mathbb{N}$ and $x, y \in X$,

$$(x <_{i[A_n]} y \wedge y <_{i[A_n^c]} x) \rightarrow x <_{\sigma(i)} y.$$

Definability lemma for σ-decisiveness

The following is provable in RCA_0. Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society. Then there exists $g : \mathbb{N} \rightarrow \mathbb{N}$ and $x, y \in X$ such that the following are equivalent for all $n \in \mathbb{N}$.

1. A_n is σ-decisive.
2. A_n is almost σ-decisive.
3. There exist $k \in \mathbb{N}$ and $x, y \in X$ such that A_n is almost σ-decisive for k at x, y.

Almost σ-decisive coalitions

We say that A_n is *almost σ-decisive for x, y at i* if

$$x <_{i[A_n]} y \wedge y <_{i[A_n^c]} x \wedge x <_{\sigma(i)} y,$$

and A_n is *almost σ-decisive* if for all $i \in \mathbb{N}$ and $x, y \in X$,

$$(x <_{i[A_n]} y \wedge y <_{i[A_n^c]} x) \rightarrow x <_{\sigma(i)} y.$$

Definability lemma for σ-decisiveness

The following is provable in RCA_0. Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society. Then there exists $g : \mathbb{N} \rightarrow \mathbb{N}$ and $x, y \in X$ such that the following are equivalent for all $n \in \mathbb{N}$.

1. A_n is σ-decisive.
2. A_n is almost σ-decisive.
3. There exist $k \in \mathbb{N}$ and $x, y \in X$ such that A_n is almost σ-decisive for k at x, y.
4. $x <_{\sigma(g(n))} y$.

KS in RCA_0

By the definability lemma for σ-decisiveness, the set

$$\mathcal{U}_\sigma = \{n : A_n \text{ is } \sigma\text{-decisive}\}$$

exists in RCA_0.

KS in RCA_0

By the definability lemma for σ-decisiveness, the set

$$\mathcal{U}_\sigma = \{n : A_n \text{ is } \sigma\text{-decisive}\}$$

exists in RCA_0.

Formalising the remainder of Kirman and Sondermann's original proof in RCA_0 we can show that

KS in RCA_0

By the definability lemma for σ-decisiveness, the set

$$\mathcal{U}_\sigma = \{n : A_n \text{ is } \sigma\text{-decisive}\}$$

exists in RCA_0.

Formalising the remainder of Kirman and Sondermann's original proof in RCA_0 we can show that

i. $\mathcal{U}_\sigma$ is an ultrafilter on $\mathcal{A}$

KS in RCA_0

By the definability lemma for σ-decisiveness, the set

$$\mathcal{U}_\sigma = \{n : A_n \text{ is } \sigma\text{-decisive}\}$$

exists in RCA_0.

Formalising the remainder of Kirman and Sondermann's original proof in RCA_0 we can show that

i. $\mathcal{U}_\sigma$ is an ultrafilter on $\mathcal{A}$, and

ii. $\mathcal{U}_\sigma$ is principal if and only if σ is dictatorial.

KS in RCA_0

By the definability lemma for σ-decisiveness, the set

$$\mathcal{U}_\sigma = \{n : A_n \text{ is } \sigma\text{-decisive}\}$$

exists in RCA_0.

Formalising the remainder of Kirman and Sondermann's original proof in RCA_0 we can show that

i. $\mathcal{U}_\sigma$ is an ultrafilter on $\mathcal{A}$, and

ii. $\mathcal{U}_\sigma$ is principal if and only if σ is dictatorial.

Theorem

KS *is provable in* RCA_0.

Arrow's theorem in RCA_0

Since all ultrafilters on a finite set V are principal, we have the following as a corollary of the RCA_0-provability of KS.

Arrow's theorem in RCA_0

Since all ultrafilters on a finite set V are principal, we have the following as a corollary of the RCA_0-provability of KS.

Theorem

Arrow's theorem is provable in RCA_0.

Arrow's theorem in RCA_0

Since all ultrafilters on a finite set V are principal, we have the following as a corollary of the RCA_0-provability of KS.

Theorem

Arrow's theorem is provable in RCA_0.

Since Arrow's theorem concerns finite societies, we can code societies and social welfare functions by natural numbers and thereby obtain a Π^0_1 formalisation θ of Arrow's theorem.

Arrow's theorem in RCA_0

Since all ultrafilters on a finite set V are principal, we have the following as a corollary of the RCA_0-provability of KS.

Theorem

Arrow's theorem is provable in RCA_0.

Since Arrow's theorem concerns finite societies, we can code societies and social welfare functions by natural numbers and thereby obtain a Π^0_1 formalisation θ of Arrow's theorem.

θ follows from the second-order formalisation in RCA_0, so $\mathrm{PRA} \vdash \theta$ by the Π^0_2 conservativity of RCA_0 over PRA.

Arrow's theorem in RCA_0

Since all ultrafilters on a finite set V are principal, we have the following as a corollary of the RCA_0-provability of KS.

Theorem

Arrow's theorem is provable in RCA_0.

Since Arrow's theorem concerns finite societies, we can code societies and social welfare functions by natural numbers and thereby obtain a Π^0_1 formalisation θ of Arrow's theorem.

θ follows from the second-order formalisation in RCA_0, so $\mathrm{PRA} \vdash \theta$ by the Π^0_2 conservativity of RCA_0 over PRA.

Conjecture. θ is provable in $\mathrm{I}\Delta_0 + \exp$.

5. The strength of Fishburn's possibility theorem

Non-dictatoriality conditions

Definition

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society and $\sigma : \mathbb{N} \to W$ is a social welfare function for $\mathcal{S}$.

Non-dictatoriality conditions

Definition

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society and $\sigma : \mathbb{N} \to W$ is a social welfare function for $\mathcal{S}$.

i. σ is *k-non-dictatorial* if for all $s \in V^{<\mathbb{N}}$ with $|s| \leq k$ there exist $j \in \mathbb{N}$ and $x, y \in X$ such that for all $i < |s|$, $x <_{j(s(i))} y$ and $y \lesssim_{\sigma(j)} x$.

Non-dictatoriality conditions

Definition

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society and $\sigma : \mathbb{N} \to W$ is a social welfare function for $\mathcal{S}$.

i. σ is *k-non-dictatorial* if for all $s \in V^{<\mathbb{N}}$ with $|s| \leq k$ there exist $j \in \mathbb{N}$ and $x, y \in X$ such that for all $i < |s|$, $x <_{j(s(i))} y$ and $y \lesssim_{\sigma(j)} x$.

ii. σ is *finitely non-dictatorial* if it is k-non-dictatorial for all k.

Non-dictatoriality conditions

Definition

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society and $\sigma : \mathbb{N} \to W$ is a social welfare function for $\mathcal{S}$.

i. σ is *k-non-dictatorial* if for all $s \in V^{<\mathbb{N}}$ with $|s| \leq k$ there exist $j \in \mathbb{N}$ and $x, y \in X$ such that for all $i < |s|$, $x <_{j(s(i))} y$ and $y \lesssim_{\sigma(j)} x$.

ii. σ is *finitely non-dictatorial* if it is k-non-dictatorial for all k.

iii. σ has the *cofinite coalitions property* if for every f_j, if cofinitely many $v \in V$ are such that $x <_{j(v)} y$, then $x <_{\sigma(j)}$ y.

Non-dictatoriality conditions

Definition

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society and $\sigma : \mathbb{N} \to W$ is a social welfare function for $\mathcal{S}$.

i. σ is *k-non-dictatorial* if for all $s \in V^{<\mathbb{N}}$ with $|s| \leq k$ there exist $j \in \mathbb{N}$ and $x, y \in X$ such that for all $i < |s|$, $x <_{j(s(i))} y$ and $y \lesssim_{\sigma(j)} x$.

ii. σ is *finitely non-dictatorial* if it is k-non-dictatorial for all k.

iii. σ has the *cofinite coalitions property* if for every f_j, if cofinitely many $v \in V$ are such that $x <_{j(v)} y$, then $x <_{\sigma(j)}$ y.

FPT^k, $\mathrm{FPT}^{<\mathbb{N}}$, and FPT^+ are the statements obtained from FPT by replacing the condition of non-dictatoriality with the conditions of k-non-dictatoriality (for some fixed $k \geq 1$), finite non-dictatoriality, and the cofinite coalitions property respectively.

The strength of FPT

Theorem

The following are equivalent over RCA_0.

The strength of FPT

Theorem

The following are equivalent over RCA_0.

1. FPT.

The strength of FPT

Theorem

The following are equivalent over RCA_0.

1. FPT.
2. FPT^k *for any* $k \geq 1$.

The strength of FPT

Theorem

The following are equivalent over RCA_0.

1. FPT.
2. FPT^k *for any* $k \geq 1$.
3. $\mathrm{FPT}^{<\mathbb{N}}$.

The strength of FPT

Theorem

The following are equivalent over RCA_0.

1. FPT.
2. FPT^k *for any* $k \geq 1$.
3. $\mathrm{FPT}^{<\mathbb{N}}$.
4. FPT^+.

The strength of FPT

Theorem

The following are equivalent over RCA_0.

1. FPT.
2. FPT^k *for any* $k \geq 1$.
3. $\mathrm{FPT}^{<\mathbb{N}}$.
4. FPT^+.
5. ACA_0.

The strength of FPT

Theorem

The following are equivalent over RCA_0.

1. FPT.
2. FPT^k *for any* $k \geq 1$.
3. $\mathrm{FPT}^{<\mathbb{N}}$.
4. FPT^+.
5. ACA_0.

The equivalence of 1, 2, 3, and 4 uses Σ^0_1 induction on the bounds of finite coalitions to show that any non-dictatorial σ has the cofinite coalitions property.

The strength of FPT

Theorem

The following are equivalent over RCA_0.

1. FPT.
2. FPT^k *for any* $k \geq 1$.
3. $\mathrm{FPT}^{<\mathbb{N}}$.
4. FPT^+.
5. ACA_0.

The equivalence of 1, 2, 3, and 4 uses Σ^0_1 induction on the bounds of finite coalitions to show that any non-dictatorial σ has the cofinite coalitions property.

All that remains is to show in RCA_0 that FPT implies ACA_0, and that FPT can be proved in ACA_0.

Non-principal ultrafilters and ACA_0

The following well-known result lies at the heart of the equivalence between ACA_0 and FPT.

Non-principal ultrafilters and ACA_0

The following well-known result lies at the heart of the equivalence between ACA_0 and FPT.

Lemma

The following are equivalent over RCA_0.

1. *For every countable atomic algebra* $\mathcal{A}$ *over an infinite set* V, *there exists a non-principal ultrafilter* $\mathcal{U}$ *on* $\mathcal{A}$.
2. ACA_0.

Reversing FPT to ACA_0

Working in $\mathsf{RCA}_0 + \text{FPT}$, fix a countably infinite atomic algebra $\mathcal{A}$.

Reversing FPT to ACA_0

Working in $\mathsf{RCA}_0 + \mathrm{FPT}$, fix a countably infinite atomic algebra $\mathcal{A}$.

Lemma

The following is provable in RCA_0. *Suppose* $V \subseteq \mathbb{N}$ *is nonempty and* $X \subseteq \mathbb{N}$ *is finite with* $|X| \geq 3$ *and* $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ *is a countable algebra over* V.

Reversing FPT to ACA_0

Working in $\mathsf{RCA}_0 + \mathrm{FPT}$, fix a countably infinite atomic algebra $\mathcal{A}$.

Lemma

The following is provable in RCA_0. Suppose $V \subseteq \mathbb{N}$ is nonempty and $X \subseteq \mathbb{N}$ is finite with $|X| \geq 3$ and $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ is a countable algebra over V. Then there exists a sequence $\mathcal{F} = \langle f_i : i \in \mathbb{N} \rangle$ of profiles over V, X such that $\mathcal{F}$ is uniformly $\mathcal{A}$-measurable and $\mathcal{A}$ is quasi-partition embedded into $\mathcal{F}$.

Reversing FPT to ACA_0

Working in $\mathsf{RCA}_0 + \mathrm{FPT}$, fix a countably infinite atomic algebra $\mathcal{A}$.

Lemma

The following is provable in RCA_0. Suppose $V \subseteq \mathbb{N}$ is nonempty and $X \subseteq \mathbb{N}$ is finite with $|X| \geq 3$ and $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ is a countable algebra over V. Then there exists a sequence $\mathcal{F} = \langle f_i : i \in \mathbb{N} \rangle$ of profiles over V, X such that $\mathcal{F}$ is uniformly $\mathcal{A}$-measurable and $\mathcal{A}$ is quasi-partition embedded into $\mathcal{F}$.

Let $V = \mathbb{N}$ and $X = 3$.

Reversing FPT to ACA_0

Working in $\mathsf{RCA}_0 + \mathrm{FPT}$, fix a countably infinite atomic algebra $\mathcal{A}$.

Lemma

The following is provable in RCA_0. Suppose $V \subseteq \mathbb{N}$ is nonempty and $X \subseteq \mathbb{N}$ is finite with $|X| \geq 3$ and $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ is a countable algebra over V. Then there exists a sequence $\mathcal{F} = \langle f_i : i \in \mathbb{N} \rangle$ of profiles over V, X such that $\mathcal{F}$ is uniformly $\mathcal{A}$-measurable and $\mathcal{A}$ is quasi-partition embedded into $\mathcal{F}$.

Let $V = \mathbb{N}$ and $X = 3$. By the lemma, there exists a countably infinite society $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$.

Reversing FPT to ACA_0

Working in $\mathsf{RCA}_0 + \text{FPT}$, fix a countably infinite atomic algebra $\mathcal{A}$.

Lemma

The following is provable in RCA_0. Suppose $V \subseteq \mathbb{N}$ is nonempty and $X \subseteq \mathbb{N}$ is finite with $|X| \geq 3$ and $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ is a countable algebra over V. Then there exists a sequence $\mathcal{F} = \langle f_i : i \in \mathbb{N} \rangle$ of profiles over V, X such that $\mathcal{F}$ is uniformly $\mathcal{A}$-measurable and $\mathcal{A}$ is quasi-partition embedded into $\mathcal{F}$.

Let $V = \mathbb{N}$ and $X = 3$. By the lemma, there exists a countably infinite society $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$. By FPT there exists a non-dictatorial social welfare function σ for $\mathcal{S}$

Reversing FPT to ACA_0

Working in $\mathsf{RCA}_0 + \text{FPT}$, fix a countably infinite atomic algebra $\mathcal{A}$.

Lemma

The following is provable in RCA_0. *Suppose* $V \subseteq \mathbb{N}$ *is nonempty and* $X \subseteq \mathbb{N}$ *is finite with* $|X| \geq 3$ *and* $\mathcal{A} = \langle A_i : i \in \mathbb{N} \rangle$ *is a countable algebra over* V. *Then there exists a sequence* $\mathcal{F} = \langle f_i : i \in \mathbb{N} \rangle$ *of profiles over* V, X *such that* $\mathcal{F}$ *is uniformly* $\mathcal{A}$*-measurable and* $\mathcal{A}$ *is quasi-partition embedded into* $\mathcal{F}$.

Let $V = \mathbb{N}$ and $X = 3$. By the lemma, there exists a countably infinite society $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$. By FPT there exists a non-dictatorial social welfare function σ for $\mathcal{S}$, and by KS there exists an ultrafilter $\mathcal{U}_\sigma$ on $\mathcal{A}$ which is non-dictatorial. □

Proving FPT in ACA_0

Lemma

The following is provable in RCA_0. *Suppose* $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ *is a countable society and* $\mathcal{U}$ *is an ultrafilter on* $\mathcal{A}$.

Proving FPT in ACA_0

Lemma

The following is provable in RCA_0*. Suppose* $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ *is a countable society and* $\mathcal{U}$ *is an ultrafilter on* $\mathcal{A}$*. Then there exists a social welfare function* $\sigma_{\mathcal{U}} : \mathbb{N} \to W$ *such that*

$$\mathcal{U} = \{i : A_i \text{ is } \sigma_{\mathcal{U}}\text{-decisive}\}$$

Proving FPT in ACA_0

Lemma

The following is provable in RCA_0. *Suppose* $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ *is a countable society and* $\mathcal{U}$ *is an ultrafilter on* $\mathcal{A}$. *Then there exists a social welfare function* $\sigma_{\mathcal{U}} : \mathbb{N} \to W$ *such that*

$$\mathcal{U} = \{i : A_i \text{ is } \sigma_{\mathcal{U}}\text{-decisive}\}$$

and $\sigma_{\mathcal{U}}$ *is dictatorial if and only if* $\mathcal{U}$ *is principal.*

Proving FPT in ACA_0

Lemma

The following is provable in RCA_0. *Suppose* $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ *is a countable society and* $\mathcal{U}$ *is an ultrafilter on* $\mathcal{A}$. *Then there exists a social welfare function* $\sigma_{\mathcal{U}} : \mathbb{N} \to W$ *such that*

$$\mathcal{U} = \{i : A_i \text{ is } \sigma_{\mathcal{U}}\text{-decisive}\}$$

and $\sigma_{\mathcal{U}}$ *is dictatorial if and only if* $\mathcal{U}$ *is principal.*

- This lemma is a partial converse to KS

Proving FPT in ACA_0

Lemma

The following is provable in RCA_0*. Suppose* $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ *is a countable society and* $\mathcal{U}$ *is an ultrafilter on* $\mathcal{A}$*. Then there exists a social welfare function* $\sigma_{\mathcal{U}} : \mathbb{N} \to W$ *such that*

$$\mathcal{U} = \{i : A_i \text{ is } \sigma_{\mathcal{U}}\text{-decisive}\}$$

and $\sigma_{\mathcal{U}}$ *is dictatorial if and only if* $\mathcal{U}$ *is principal.*

- This lemma is a partial converse to KS—partial because $\sigma_{\mathcal{U}}$ is in general not unique.

Proving FPT in ACA_0

Lemma

The following is provable in RCA_0. *Suppose* $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ *is a countable society and* $\mathcal{U}$ *is an ultrafilter on* $\mathcal{A}$. *Then there exists a social welfare function* $\sigma_{\mathcal{U}} : \mathbb{N} \to W$ *such that*

$$\mathcal{U} = \{i : A_i \text{ is } \sigma_{\mathcal{U}}\text{-decisive}\}$$

and $\sigma_{\mathcal{U}}$ *is dictatorial if and only if* $\mathcal{U}$ *is principal.*

- This lemma is a partial converse to KS—partial because $\sigma_{\mathcal{U}}$ is in general not unique.

Working in ACA_0, let $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ be a countably infinite society.

Proving FPT in ACA_0

Lemma

The following is provable in RCA_0*. Suppose* $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ *is a countable society and* $\mathcal{U}$ *is an ultrafilter on* $\mathcal{A}$*. Then there exists a social welfare function* $\sigma_{\mathcal{U}} : \mathbb{N} \to W$ *such that*

$$\mathcal{U} = \{i : A_i \text{ is } \sigma_{\mathcal{U}}\text{-decisive}\}$$

and $\sigma_{\mathcal{U}}$ *is dictatorial if and only if* $\mathcal{U}$ *is principal.*

- This lemma is a partial converse to KS—partial because $\sigma_{\mathcal{U}}$ is in general not unique.

Working in ACA_0, let $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ be a countably infinite society. By our earlier lemma, there exists a non-principal ultrafilter $\mathcal{U}$ on $\mathcal{A}$

Proving FPT in ACA_0

Lemma

The following is provable in RCA_0. *Suppose* $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ *is a countable society and* $\mathcal{U}$ *is an ultrafilter on* $\mathcal{A}$. *Then there exists a social welfare function* $\sigma_{\mathcal{U}} : \mathbb{N} \to W$ *such that*

$$\mathcal{U} = \{i : A_i \text{ is } \sigma_{\mathcal{U}}\text{-decisive}\}$$

and $\sigma_{\mathcal{U}}$ *is dictatorial if and only if* $\mathcal{U}$ *is principal.*

- This lemma is a partial converse to KS—partial because $\sigma_{\mathcal{U}}$ is in general not unique.

Working in ACA_0, let $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ be a countably infinite society. By our earlier lemma, there exists a non-principal ultrafilter $\mathcal{U}$ on $\mathcal{A}$, so by our partial converse to KS there exists a non-dictatorial social welfare function $\sigma_{\mathcal{U}}$ for $\mathcal{S}$. $\square$

Some open questions about FPT

Can we have a more revealing computability-theoretic analysis of non-dictatorial social welfare functions?

Some open questions about FPT

Can we have a more revealing computability-theoretic analysis of non-dictatorial social welfare functions?

i. By KS we have that $\mathcal{U}_\sigma \leq_{\mathrm{T}} \mathcal{S} \oplus \sigma$, but this is *not* in general an equivalence.

Some open questions about FPT

Can we have a more revealing computability-theoretic analysis of non-dictatorial social welfare functions?

i. By KS we have that $\mathcal{U}_\sigma \leq_{\mathrm{T}} \mathcal{S} \oplus \sigma$, but this is *not* in general an equivalence.

   At least for dictatorial σ, it is possible to have $\mathcal{U}_\sigma <_{\mathrm{T}} \sigma$, by a kind of filling-in construction. Can we also have this for non-dictatorial σ?

Some open questions about FPT

Can we have a more revealing computability-theoretic analysis of non-dictatorial social welfare functions?

i. By KS we have that $\mathcal{U}_\sigma \leq_{\mathrm{T}} \mathcal{S} \oplus \sigma$, but this is *not* in general an equivalence.

   At least for dictatorial σ, it is possible to have $\mathcal{U}_\sigma <_{\mathrm{T}} \sigma$, by a kind of filling-in construction. Can we also have this for non-dictatorial σ?

ii. A standard construction of a non-principal ultrafilter provides us with a non-dictatorial $\sigma_{\mathcal{U}}$ with PA-degree relative to $\mathcal{S}''$. Can this be reduced?

Some open questions about FPT

Can we have a more revealing computability-theoretic analysis of non-dictatorial social welfare functions?

i. By KS we have that $\mathcal{U}_\sigma \leq_{\mathrm{T}} \mathcal{S} \oplus \sigma$, but this is *not* in general an equivalence.

   At least for dictatorial σ, it is possible to have $\mathcal{U}_\sigma <_{\mathrm{T}} \sigma$, by a kind of filling-in construction. Can we also have this for non-dictatorial σ?

ii. A standard construction of a non-principal ultrafilter provides us with a non-dictatorial $\sigma_{\mathcal{U}}$ with PA-degree relative to $\mathcal{S}''$. Can this be reduced?

iii. Conversely, all our reversal does is show that a non-dictatorial σ computes $\mathcal{S}'$, via the usual Kirby construction. Can this be strengthened?

6. Coda: Strategic voting theorems

Social choice functions and the Gibbard–Satterthwaite theorem

A *social choice function* is a function $c : \mathcal{F} \to X$ which selects a single overall winner for any given election.

Social choice functions and the Gibbard–Satterthwaite theorem

A *social choice function* is a function $c : \mathcal{F} \to X$ which selects a single overall winner for any given election.

Gibbard (1973) and Satterthwaite (1975) investigated *strategic voting*

Social choice functions and the Gibbard–Satterthwaite theorem

A *social choice function* is a function $c : \mathcal{F} \to X$ which selects a single overall winner for any given election.

Gibbard (1973) and Satterthwaite (1975) investigated *strategic voting*, in which voters may misrepresent their true preferences in order to achieve an electoral outcome which they prefer to the one that would be achieved if they voted according to their true preferences.

Social choice functions and the Gibbard–Satterthwaite theorem

A *social choice function* is a function $c : \mathcal{F} \to X$ which selects a single overall winner for any given election.

Gibbard (1973) and Satterthwaite (1975) investigated *strategic voting*, in which voters may misrepresent their true preferences in order to achieve an electoral outcome which they prefer to the one that would be achieved if they voted according to their true preferences.

Gibbard–Satterthwaite theorem

If $\mathcal{S}$ is a finite society and c is a social choice function for $\mathcal{S}$ which is immune to strategic voting, then c is dictatorial.

Manipulability and strategyproofness

Definition

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society and $c : \mathbb{N} \to X$ is a social choice function for $\mathcal{S}$.

Manipulability and strategyproofness

Definition

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society and $c : \mathbb{N} \to X$ is a social choice function for $\mathcal{S}$. c is *manipulable* at f_n by v if $v \in V$ and there exists $k \in \mathbb{N}$ such that

Manipulability and strategyproofness

Definition

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society and $c : \mathbb{N} \to X$ is a social choice function for $\mathcal{S}$. c is *manipulable* at f_n by v if $v \in V$ and there exists $k \in \mathbb{N}$ such that

1. $f_k(v) \neq f_n(v)$

Manipulability and strategyproofness

Definition

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society and $c : \mathbb{N} \to X$ is a social choice function for $\mathcal{S}$. c is *manipulable* at f_n by v if $v \in V$ and there exists $k \in \mathbb{N}$ such that

1. $f_k(v) \neq f_n(v)$,
2. For all $v' \in V$ such that $v' \neq v$, $f_k(v') = f_n(v')$

Manipulability and strategyproofness

Definition

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society and $c : \mathbb{N} \to X$ is a social choice function for $\mathcal{S}$. c is *manipulable* at f_n by v if $v \in V$ and there exists $k \in \mathbb{N}$ such that

1. $f_k(v) \neq f_n(v)$,
2. For all $v' \in V$ such that $v' \neq v$, $f_k(v') = f_n(v')$,
3. $c(k) <_{n(v)} c(n)$.

Manipulability and strategyproofness

Definition

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society and $c : \mathbb{N} \to X$ is a social choice function for $\mathcal{S}$. c is *manipulable* at f_n by v if $v \in V$ and there exists $k \in \mathbb{N}$ such that

1. $f_k(v) \neq f_n(v)$,
2. For all $v' \in V$ such that $v' \neq v$, $f_k(v') = f_n(v')$,
3. $c(k) <_{n(v)} c(n)$.

If c is not manipulable at any f_n by any $v \in V$, then c is *strategyproof*.

Manipulability and strategyproofness

Definition

Suppose $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ is a countable society and $c : \mathbb{N} \to X$ is a social choice function for $\mathcal{S}$. c is *manipulable* at f_n by v if $v \in V$ and there exists $k \in \mathbb{N}$ such that

1. $f_k(v) \neq f_n(v)$,
2. For all $v' \in V$ such that $v' \neq v$, $f_k(v') = f_n(v')$,
3. $c(k) <_{n(v)} c(n)$.

If c is not manipulable at any f_n by any $v \in V$, then c is *strategyproof*.

Theorem (Pazner and Wesley 1977)

If $\mathcal{S}$ is an infinite society there exists a non-dictatorial, strategyproof social choice function c for $\mathcal{S}$.

Coalitional manipulability

Strategyproof social choice functions can be understood in terms of ultrafilters, but (individual) strategyproofness does not generalise nicely to the infinite case.

Coalitional manipulability

Strategyproof social choice functions can be understood in terms of ultrafilters, but (individual) strategyproofness does not generalise nicely to the infinite case.

Definition

Let $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ be a countable society and $c : \mathbb{N} \to X$ be a social choice function for $\mathcal{S}$.

Coalitional manipulability

Strategyproof social choice functions can be understood in terms of ultrafilters, but (individual) strategyproofness does not generalise nicely to the infinite case.

Definition

Let $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ be a countable society and $c : \mathbb{N} \to X$ be a social choice function for $\mathcal{S}$. c is *coalitionally manipulable at* f_n *by* A_m if there exists k such that for all $v \notin A_m$, $f_k(v) = f_n(v)$, and for all $v \in A_m$, $c(k) <_{n(v)} c(n)$.

Coalitional manipulability

Strategyproof social choice functions can be understood in terms of ultrafilters, but (individual) strategyproofness does not generalise nicely to the infinite case.

Definition

Let $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ be a countable society and $c : \mathbb{N} \to X$ be a social choice function for $\mathcal{S}$. c is *coalitionally manipulable at f_n by A_m* if there exists k such that for all $v \notin A_m$, $f_k(v) = f_n(v)$, and for all $v \in A_m$, $c(k) <_{n(v)} c(n)$. c is *coalitionally strategyproof* if it is not coalitionally manipulable at any f_n by any A_m.

Coalitional manipulability

Strategyproof social choice functions can be understood in terms of ultrafilters, but (individual) strategyproofness does not generalise nicely to the infinite case.

Definition

Let $\mathcal{S} = \langle V, X, \mathcal{A}, \mathcal{F} \rangle$ be a countable society and $c : \mathbb{N} \to X$ be a social choice function for $\mathcal{S}$. c is *coalitionally manipulable at f_n by A_m* if there exists k such that for all $v \notin A_m$, $f_k(v) = f_n(v)$, and for all $v \in A_m$, $c(k) <_{n(v)} c(n)$. c is *coalitionally strategyproof* if it is not coalitionally manipulable at any f_n by any A_m.

Theorem (Pazner and Wesley 1977)

If $\mathcal{S}$ is an infinite society there exists a non-dictatorial, coalitionally strategyproof social choice function c for $\mathcal{S}$.

The strength of the Pazner–Wesley possibility theorem

Theorem

The following are equivalent over RCA_0.

The strength of the Pazner–Wesley possibility theorem

Theorem

The following are equivalent over RCA_0.

1. *For every countably infinite society* $\mathcal{S}$ *there exists a non-dictatorial, coalitionally strategyproof social choice function* c *for* $\mathcal{S}$.

The strength of the Pazner–Wesley possibility theorem

Theorem

The following are equivalent over RCA_0.

1. *For every countably infinite society* $\mathcal{S}$ *there exists a non-dictatorial, coalitionally strategyproof social choice function* c *for* $\mathcal{S}$.
2. *Fishburn's possibility theorem for countable societies.*

The strength of the Pazner–Wesley possibility theorem

Theorem

The following are equivalent over RCA_0.

1. *For every countably infinite society* $\mathcal{S}$ *there exists a non-dictatorial, coalitionally strategyproof social choice function* c *for* $\mathcal{S}$.
2. *Fishburn's possibility theorem for countable societies.*
3. ACA_0.

The strength of the Pazner–Wesley possibility theorem

Theorem

The following are equivalent over RCA_0.

1. *For every countably infinite society* $\mathcal{S}$ *there exists a non-dictatorial, coalitionally strategyproof social choice function* c *for* $\mathcal{S}$.
2. *Fishburn's possibility theorem for countable societies.*
3. ACA_0.

Theorem (Mihara 2000)

There exists a countably infinite society $\mathcal{S}_M$ *and a non-dictatorial social choice function* c_M *for* $\mathcal{S}_M$ *which is individually but not coalitionally strategyproof.*

The strength of the Pazner–Wesley possibility theorem

Theorem

The following are equivalent over RCA_0.

1. *For every countably infinite society* $\mathcal{S}$ *there exists a non-dictatorial, coalitionally strategyproof social choice function* c *for* $\mathcal{S}$.
2. *Fishburn's possibility theorem for countable societies.*
3. ACA_0.

Theorem (Mihara 2000)

There exists a countably infinite society $\mathcal{S}_M$ *and a non-dictatorial social choice function* c_M *for* $\mathcal{S}_M$ *which is individually but not coalitionally strategyproof.*

Question. What is the strength of the statement that for every countably infinite society there exists an *individually* strategyproof social choice function?

Thank you!

References I

T. E. Armstrong. Arrow's theorem with restricted coalition algebras. *Journal of Mathematical Economics*, 7(1):55–75, March 1980. doi: 10.1016/0304-4068(80)90021-X.

K. J. Arrow. A difficulty in the concept of social welfare. *Journal of Political Economy*, 58(4):328–346, August 1950. doi: 10.1086/256963.

P. C. Fishburn. Arrow's impossibility theorem: Concise proof and infinite voters. *Journal of Economic Theory*, 2(1):103–106, March 1970. doi: 10.1016/0022-0531(70)90015-3.

A. Gibbard. Manipulation of voting schemes: a general result. *Econometrica*, 41(4): 587–601, July 1973. doi: 10.2307/1914083.

A. P. Kirman and D. Sondermann. Arrow's theorem, many agents, and invisible dictators. *Journal of Economic Theory*, 5(2):267–277, October 1972. doi: 10.1016/0022-0531(72)90106-8.

References II

U. Kohlenbach. Foundational and mathematical uses of higher types. In W. Sieg, R. Sommer, and C. Talcott, editors, *Reflections on the Foundations of Mathematics: Essays in Honor of Solomon Feferman*, number 15 in Lecture Notes in Logic, pages 92–116. Association for Symbolic Logic and A. K. Peters, Natick, MA, 2002. doi: 10.1017/9781316755983.005.

U. Kohlenbach. Higher order reverse mathematics. In S. G. Simpson, editor, *Reverse Mathematics 2001*, volume 21 of *Lecture Notes in Logic*, pages 281–295. A. K. Peters, 2005. doi: 10.1017/9781316755846.018.

T. Litak. Infinite populations, choice and determinacy. *Studia Logica*, 106(5): 969–999, October 2018. doi: 10.1007/s11225-017-9730-3. Special Issue: Logics for Social Behaviour, edited by Alessandra Palmigiano and Marcus Pivato.

H. R. Mihara. Arrow's theorem, countably many agents, and more visible invisible dictators. *Journal of Mathematical Economics*, 32(3):267–287, 1999. doi: 10.1016/S0304-4068(98)00061-5.

References III

H. R. Mihara. Coalitionally strategyproof functions depend only on the most-preferred alternatives. *Social Choice and Welfare*, 17(3):393–402, 2000. doi: 10.1007/s003550050170.

E. A. Pazner and E. Wesley. Stability of social choices in infinitely large societies. *Journal of Economic Theory*, 14(2):252–262, April 1977. doi: 10.1016/0022-0531(77)90129-6.

W. H. Riker. *Liberalism Against Populism: A Confrontation Between the Theory of Democracy and the Theory of Social Choice*. Waveland Press, Long Grove, IL, 1982.

M. Satterthwaite. Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10(2):187–217, April 1975. doi: 10.1016/0022-0531(75)90050-2.

H. Towsner. Ultrafilters in reverse mathematics. *Journal of Mathematical Logic*, 14 (1):1–11, 2014. doi: 10.1142/S0219061314500019.