THE STRENGTH OF THE EXISTENCE OF POSITIVE, FRIEDBERG AND MINIMAL NUMBERINGS

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Numberings

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Elements of $\mathcal{S}$ could be:

- formulae of first order arithmetic (Gödel)
- partial computable functions
- recursive models
- finite sets
- tuples of $\mathbb{N}^n$

Generalised approach of Goncharov and Sorbi

Consider families of objects which admits a constructive description in a certain formal language.
Elements of $\mathcal{S}$ could be:

- Σ^0_n-formulae of first order arithmetic
- partial Σ^0_n-computable functions
- Δ^0_n-recursive models

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- Σ^0_n-formulae of first order arithmetic
- partial Σ^0_n-computable functions
- Δ^0_n-recursive models

Let $\mathcal{S}$ be at most countable family of Σ^0_n-subsets of $\mathbb{N}$.
A numbering of $\nu\colon \mathbb{N} \to \mathcal{S}$ is Σ^0_n-computable if $\{\langle m, x\rangle \mid x \in \nu(m)\} \in \Sigma^0_n$.

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ν is reducible to μ, $\nu \leq \mu$, if there exists a total computable function $\mathrm{f}\colon \mathbb{N} \to \mathbb{N}$ such that $\nu(\mathrm{k}) = \mu(\mathrm{f}(\mathrm{k}))$ for all $\mathrm{k} \in \mathbb{N}$.

Numberings ν and μ are equivalent if $\nu \leq \mu$ and $\mu \leq \nu$.

Interesting numberings

A numbering $\nu \colon \mathbb{N} \to \mathcal{S}$ is called Friedberg if ν is injective, i.e. $\nu(k) \neq \nu(l)$ for all $k \neq l$.

A numbering $\nu \colon \mathbb{N} \to \mathcal{S}$ is positive if the set $\{\langle k, l \rangle : \nu(k) = \nu(l)\}$ is c.e.

A numbering $\nu \colon \mathbb{N} \to \mathcal{S}$ is minimal if for every numbering $\mu \colon \mathbb{N} \to \mathcal{S}$ it holds that if $\mu \leq \nu$, then $\mu \equiv \nu$.

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Friedberg $\Rightarrow$ positive $\Rightarrow$ minimal

Wei Li's analysis

Theorem (Li)

Over $\mathrm{PA}^- + \mathrm{B}\Sigma_2$,

$\mathrm{I}\Sigma_2$ is equivalent to the existence (an index for) a Friedberg numbering for the family of all c.e. sets

Wei Li, Friedberg numbering in fragments of Peano arithmetic and α-recursion theory, 2013

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Let $\langle W_0, W_1, \ldots \rangle$ be an enumeration of all c.e. sets.

Use $\mathrm{I}\Sigma_2$ to argue that, for each $e \in \mathbb{N}$, there exists the least index i for W_e, that is

$$\forall j < i \, (W_i \neq W_j)$$

Define a Friedberg numbering for $\langle W_0, W_1, \ldots \rangle$

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Numberings in RM

Σ^0_n-numbering ν is coded by a Σ^0_n-formula $\psi(x, y)$, that is

$$\nu(x) = \{y : \psi(x, y)\}, \text{ for } x \in \mathbb{N}.$$

Theorem

Let $n \neq 0$. Over RCA_0 the following are equivalent:

- ACA_0
- for any $A \subseteq \mathbb{N}$ and any Σ^A_n-numbering ν there exists a set $B \subseteq \mathbb{N}$ and a Friedberg Σ^B_n-numbering μ such that ν and μ index the same family

• $\mathsf{ACA}_0 \vdash$ for any Σ^A_n-numbering ν there exists a Friedberg Σ^B_n-numbering •

Let $\psi(x, y, A)$ be a Σ^0_n-formula encoding $\nu\colon \mathbb{N} \to \mathcal{S}$.

$$X = \{e : (\forall i < e)(\nu(i) \neq \nu(e))\} \in \Sigma^{A'}_n$$

By $\mathsf{RCA}_0 + \mathrm{I}\Sigma^0_n$, let $\pi_X\colon \mathbb{N} \to \mathbb{N}$ be an injective $\Delta^{A'}_n$-enumeration of X.

The Friedberg numbering μ is defined as follows:

$$\xi(x, y, A') = \exists z(\pi_X(x) = z \wedge \psi(z, y, A'))$$

Then μ is a Friedberg $\Sigma^{A'}_n$-numbering of the family $\mathcal{S}$

• for any Σ^A_n-numbering ν there exists a Friedberg Σ^B_n-numbering $\vdash$ ACA_0 •

Let $g\colon \mathbb{N} \to \mathbb{N}$ be injective.
We define a Σ^0_1-numbering ν of a family $\mathcal{S}$ as follows:

$$\nu(2x) = \{2x\} \cup \{2x+1 : \exists t\,(g(t) = x)\},$$
$$\nu(2x+1) = \{2x+1\} \cup \{2x : \exists t\,(g(t) = x)\}.$$

- $x \notin \mathrm{range}(g) \Rightarrow \nu(2x) = \{2x\}$ and $\nu(2x+1) = \{2x+1\}$
- $x \in \mathrm{range}(g) \Rightarrow \nu(2x) = \nu(2x+1) = \{2x, 2x+1\}$

Let μ be a Friedberg Σ^0_1-numbering of $\mathcal{S}$. Then the set $\mathrm{range}(g)$ is Δ^0_1-definable:

$$x \notin \mathrm{range}(g) \;\Leftrightarrow\; \exists i \exists j [i \neq j \wedge 2x \in \mu(i) \wedge 2x+1 \in \mu(j)]$$

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$$\forall k\big[\nu(k) = \mu(\varphi_e(k))\big]$$

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Numberings ν and μ are equivalent if $\nu \leq \mu$ and $\mu \leq \nu$.

A Σ^0_n-numbering $\nu\colon \mathbb{N} \to \mathcal{S}$ is minimal if for any Σ^0_n-numbering μ of $\mathcal{S}$ $\mu \leq \nu$ implies $\nu \leq \mu$.

Theorem

Suppose that $n \geq 2$. Over RCA_0, $I\Sigma_2$ proves the following principle: for any Σ^A_n-numbering ν there exists a minimal Σ^A_n-numbering μ such that ν and μ index the same family.

Badaev Goncharov, Rogers semilattices of families of arithmetic sets , 2001

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Theorem (Chong Yang)

Over $RCA_0 + B\Sigma_2$, $I\Sigma_2$ is equivalent to the existence of a maximal Σ_1-set.

Badaev Goncharov, Rogers semilattices of families of arithmetic sets , 2001

Positive numberings

A Σ^0_n-numbering $\nu\colon \mathbb{N} \to \mathcal{S}$ is positive if there exists a Σ_1-formula $\theta(x, y)$ such that:

$$\forall x \forall y [\nu(x) = \nu(y) \leftrightarrow \theta(x, y)]$$

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Theorem

Let $n \in \mathbb{N}$. Over $\mathsf{RCA}_0 + \mathrm{I}\Sigma_2$ the following are equivalent:

- ACA_0
- for any Σ^A_n-numbering ν there exists a positive Σ^B_n-numbering μ such that ν and μ index the same family.

existence of Σ_n^0-Friedberg

existence of Σ_n^0-positive

existence of Σ_n^0-minimal

existence of Σ^0_n-Friedberg

ACA_0

existence of Σ^0_n-positive

$\mathsf{RCA}_0 + \mathrm{I}\Sigma_2$

existence of Σ^0_n-minimal

existence of Σ^0_n-Friedberg

ACA_0 internal properties

existence of Σ^0_n-positive

$RCA_0 + I\Sigma_2$ existence of Σ^0_n-minimal global properties

Weihrauch reducibility

A Σ^0_n-coded numbering is a sequence $p \in \omega^\omega$ satisfying the followings:

- $p = p_1 \oplus p_2$,
- for each $k \in \omega$, $p_1(k) = \langle x, y, \sigma \rangle$ for some $x, y \in \omega$ and $\sigma \in \omega^{<\omega}$,
- for any $x, y \in \omega$, $\exists k \, \exists \sigma \, p_1(k) = \langle x, y, \sigma \rangle \Rightarrow \forall \tau \sqsupseteq \sigma \, \exists m \, p_1(m) = \langle x, y, \tau \rangle$,
- p_2 is a Δ^0_n-presentation of a function $g \in \omega^\omega$,
  i.e. $g(a) = \lim_{s_1} \dots \lim_{s_{n-1}} p_2(\langle a, s_1, \dots, s_{n-1} \rangle)$ for each $a \in \omega$.

A numbering ν is represented by a Σ^0_n-coded numbering $p = p_1 \oplus p_2$ (where p_2 is a Δ^0_n-presentation of a function g) if and only if for every $x \in \omega$,

$$\nu(x) = \{y \in \omega : \exists k \, \exists \sigma \sqsubset g \, (p_1(k) = \langle x, y, \sigma \rangle)\}$$

Friedberg numberings

Let $n \geq 1$. Then $\mathrm{Fried}_n \colon \subseteq \omega^\omega \rightrightarrows \omega^\omega$ is the following multi-valued function:

- Input/instance: a Σ^0_n-coded numbering ν of an infinite family,
- Output/solution: a Σ^0_1-coded Friedberg numbering μ of the same family.

$\lim \colon \omega^{\omega \times \omega} \to \omega^\omega$ is the following function.

- Input/instance: An $f \in \omega^{\omega \times \omega}$ such that $\lim f$ exists,
- Output/solution: $\lim f$.

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Theorem

$\lim^{(n-1)} \equiv_{\mathrm{sW}} \mathrm{Fried}_n$.