Ordinal Suprema and Second Order Arithmetic

Marcia Groszek

joint work with Ben Logsdon and Justin Miller

Dartmouth College

Computability and Combinatorics 2023

Outline

Context

Reverse Mathematics: Calibrate logical strength of theorems by set-theoretic existence axioms.

Use a first-order theory of second-order arithmetic.

RCA: P^- (finitary part of Peano Arithmetic), induction for all formulas, recursive (Δ^0_1) comprehension axiom.

RCA_0: Weaken induction to Σ^0_1 formulas.

RCA^*_0: Weaken induction to Δ^0_1 formulas; exponentiation is total.

Models of RCA_0^*

Factorization of polynomials and Σ_1^0 induction

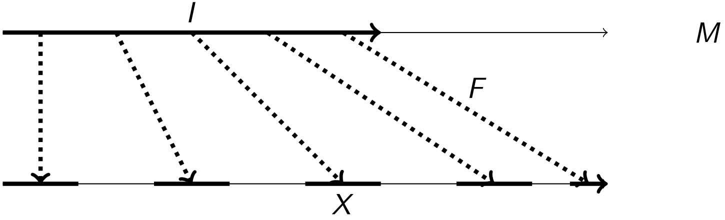
Stephen G. Simpson and Rick L. Smith

1986

Annals of Pure and Applied Logic

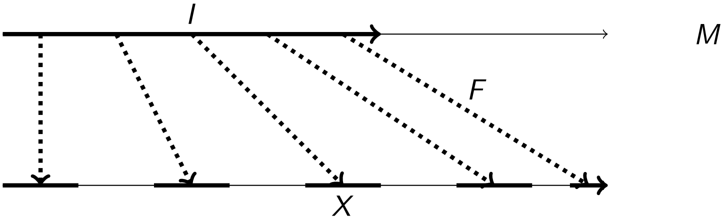
A model M of $\mathrm{RCA}_0^* + \neg I\Sigma_1^0$ has Σ_1^0-definable proper cuts.

I is Σ_1^0-definable but not an element of M.



F is increasing and cofinal with range X.

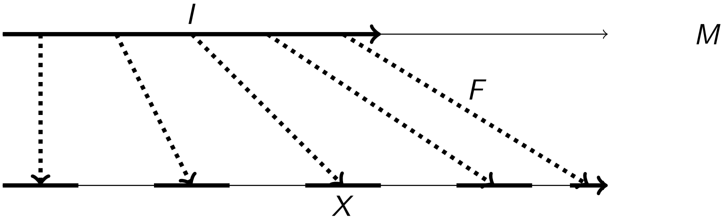
F and X are elements of M.



You can also get this picture by starting with:

- An unbounded set X in M to enumerate in order.
- A definition by primitive recursion of an increasing function F.

Cardinality in RCA_0^*



If I is proper, X is a subcountable infinite set.

Cardinals in M: Numbers, proper Σ^0_1-definable cuts, ω_M.
The cardinals are linearly ordered.

Cardinality behaves well with respect to functions between sets, pairwise sums, and pairwise products.

Let I be a Σ^0_1-definable cut closed under addition.

(There are many in any model of $RCA^*_0 + \neg I\Sigma^0_1$.)

I is an (additively) indecomposable cardinal:

$$(\kappa < I) \,\&\, (\lambda < I) \implies (\kappa + \lambda < I).$$

If $\kappa < I$ and μ is any cardinal,

$$(CARD(A \cup B) = \mu + I) \,\&\, (CARD(A) = \kappa) \implies$$
$$CARD(B) = \mu + I.$$

Ordinals in RCA_0

A survey of the reverse mathematics of ordinal arithmetic

Jeffry L. Hirst

2005

Reverse Mathematics 2001 ed. S. G. Simpson

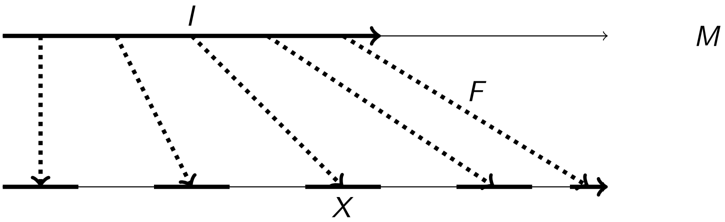
A linear ordering $\langle X, \leq_X \rangle$ is ill-founded iff there is (equivalently):

- A nonempty $Y \subseteq X$ with no $\leq_X$-least element.
- A Σ^0_1-definable cut I with a decreasing function $F : I \to X$.

An ordinal is a well-founded linear ordering $\langle \alpha, \leq_\alpha \rangle$.

The ordinals are closed under pairwise addition.

For any Σ^0_1-definable cut I we have:



The set X with the usual ordering is an ordinal of order type I:

$$\alpha_I = \langle X, \leq \rangle.$$

If I is closed under addition, α_I is (additively) indecomposable.

Comparing ordinals

For ordinals α and β define

$\alpha \leq_w \beta$ iff there is an order-preserving embedding $F : \alpha \to \beta$,

$\alpha <_w \beta$ iff $\alpha + 1 \leq_w \beta$.

Theorem (RCA_0^*)

For ordinals α and β,

$$CARD(\alpha) < CARD(\beta) \implies \alpha <_w \beta.$$

Theorem (RCA_0; H. Friedman and J. Hirst)

$ATR_0 \iff$ *all ordinals are $\leq_w$-comparable.*

Theorem ($RCA_0^* + \neg I\Sigma_1^0$)

There are ordinals α and β such that

$$\alpha \not\leq_w \beta \quad \& \quad \beta \not\leq_w \alpha.$$

Corollary

Friedman and Hirst's theorem holds over the base theory RCA_0^*.

Theorem ($RCA_0^* + \neg I\Sigma_1^0$)
There are ordinals α and β such that

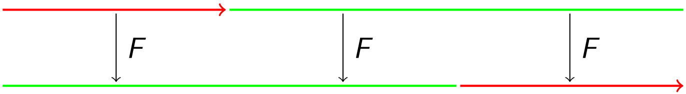
$$\alpha \not\leq_w \beta \quad \& \quad \beta \not\leq_w \alpha.$$

Proof: Let I be a Σ_1^0-definable cut closed under addition, and b a number bounding I.

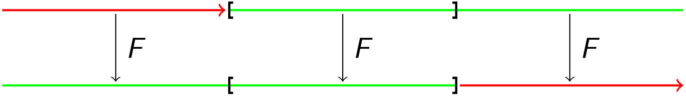
That is, $I < b$ as cardinals.

Define ordinals $\alpha = \alpha_I + b$ and $\beta = b + \alpha_I$.

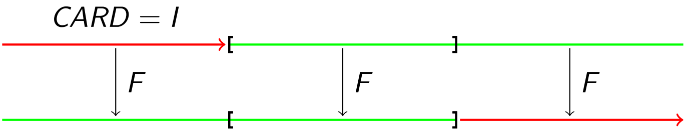
To show $\alpha \not\leq_w \beta$, suppose $F : \alpha \to \beta$ is an embedding.



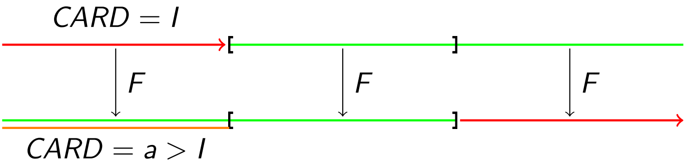
To show $\alpha \not\leq_w \beta$, suppose $F : \alpha \to \beta$ is an embedding.



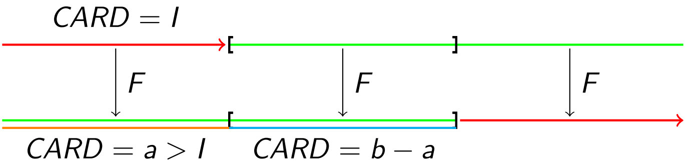
To show $\alpha \not\leq_w \beta$, suppose $F : \alpha \to \beta$ is an embedding.



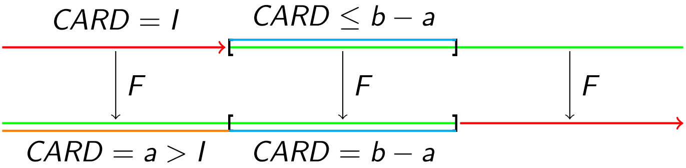
To show $\alpha \not\leq_w \beta$, suppose $F : \alpha \to \beta$ is an embedding.



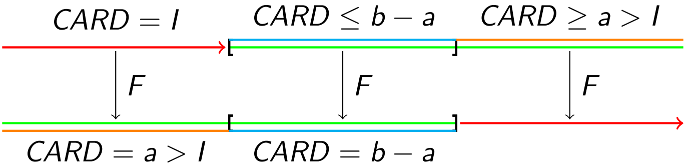
To show $\alpha \not\leq_w \beta$, suppose $F : \alpha \to \beta$ is an embedding.



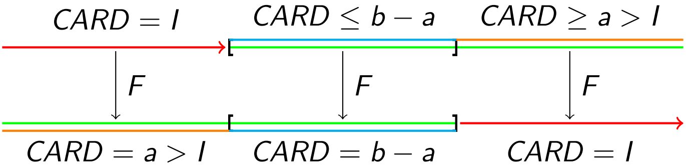
To show $\alpha \not\leq_w \beta$, suppose $F : \alpha \to \beta$ is an embedding.



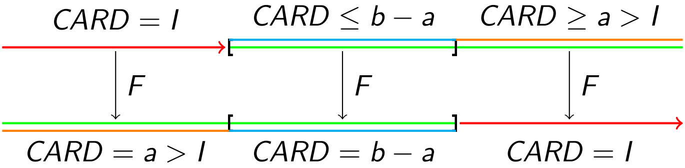
To show $\alpha \not\leq_w \beta$, suppose $F : \alpha \to \beta$ is an embedding.



To show $\alpha \not\leq_w \beta$, suppose $F : \alpha \to \beta$ is an embedding.

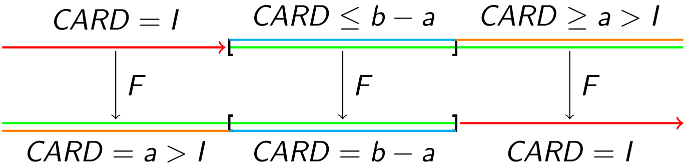


To show $\alpha \not\leq_w \beta$, suppose $F : \alpha \to \beta$ is an embedding.



This is a contradiction.

To show $\alpha \not\leq_w \beta$, suppose $F : \alpha \to \beta$ is an embedding.



This is a contradiction.

The proof $\beta \not\leq_w \alpha$ is similar.

Ordinal suprema

If $S = \langle \beta_x \mid x \in \alpha \rangle$ is a well-ordered sequence of ordinals, the ordinal γ is:

An upper bound for S if $(\forall x \in \alpha)\,(\beta_x \leq_w \gamma)$.

A minimal upper bound for S if γ is an upper bound and no proper initial segment of γ is an upper bound.

A supremum for S if γ is the unique minimal upper bound up to isomorphism.

Theorem (RCA_0; J. Hirst)

ATR_0 iff every well-ordered sequence of ordinals has a supremum.

Show RCA_0^* suffices as a base theory.

Find a weaker form of "every well-ordered sequence of ordinals has a supremum" that is equivalent to $I\Sigma_1^0$.

Fact (RCA_0^*)

If γ is a minimal upper bound for $S = \langle \beta_x \mid x \in \alpha \rangle$, then $CARD(\gamma)$ is the least cardinal κ such that

$$(\forall x \in \alpha)\,(CARD(\beta_x) \leq \kappa).$$

Corollary

Suppose κ is an infinite cardinal, every β_x has cardinality less than κ, and the cardinals $CARD(\beta_x)$ are unbounded in κ.

Then γ is a minimal upper bound for S iff $CARD(\gamma) = \kappa$ and every proper initial segment of γ has cardinality less than κ.

Corollary

The ordinal γ is a minimal upper bound for $\langle n \mid n \in \omega_M \rangle$ iff γ is countable and every proper initial segment of γ is subcountable.

Theorem (RCA_0^*)

TFAE

1. ACA_0.
2. *Every well-ordered sequence of finite ordinals has a supremum.*
3. $\langle n \mid n \in \omega_M \rangle$ *has a supremum.*

Key to proof: $ACA_0 \iff \omega_M$ is the unique countable ordinal all of whose proper initial segments are subcountable.

Corollary

The theory RCA_0^* suffices as the base theory in Hirst's theorem that ATR_0 holds iff every well-ordered sequence of ordinals has a supremum.

Theorem (RCA_0^*)

TFAE

1. $B\Sigma^0_2$.
2. *Every finite-length sequence of finite ordinals has a supremum.*

Key idea for (2) $\implies$ (1): By a result of Hirst, if $B\Sigma^0_2$ fails there is a coloring of ω_M in finitely many colors such that each color class is finite. RCA_0^* suffices to prove this.

The sequence of color classes, each ordered via the natural ordering, has multiple minimal upper bounds: ω_M, and any other countable ordinal all of whose proper initial segments are subcountable.

Theorem (RCA_0^*)

TFAE

1. $I\Sigma^0_1$.
2. *Every finite-length sequence of finite ordinals whose sizes have a finite upper bound has a supremum.*

Key idea for (2) $\implies$ (1): Suppose $I\Sigma^0_1$ fails, and let I be a non-minimal indecomposable Σ^0_1-definable cut, $F : I \to \omega_M$ be a function in M, and a be a number bounding I.

Define $S = \langle \beta_k \mid k < a \rangle$ where $\beta_k = \{x + m \mid x = F(k) \,\&\, m < k\}$ with the natural ordering. The sizes of the β_k are bounded by I, and by a.

There are multiple minimal upper bounds for S: α_I, and $\alpha_J + \alpha_I$ for any Σ^0_1-definable cut $J < I$.

Existence of minimal upper bounds

Theorem (RCA_0; J. Hirst)

ATR_0 iff every well-ordered sequence of ordinals has a supremum.

This, as well as the preceding theorems, relies on examples of sequences that do not have suprema because they have multiple minimal upper bounds.

Theorem (RCA_0^*)

If $I\Sigma_1^0$ does not hold, there is a (finite) sequence of (subcountable) ordinals with no minimal upper bound.

Proof: Let I be an indecomposable Σ_1^0-definable cut, and b and d be numbers bounding I.

For $n \leq 2d$, let $\beta_n = nb + \alpha_I + (2d - n)b$, with the natural ordering.

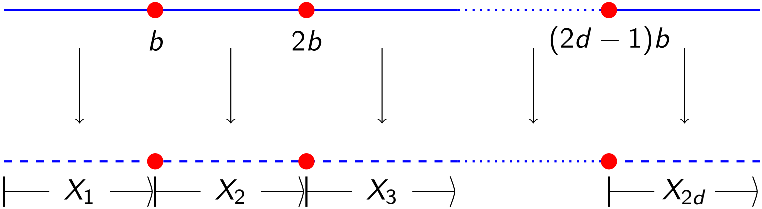
For every n we have $CARD(\beta_n) = 2db + I$.

Let $S = \langle \beta_n \mid n \leq 2d \rangle$.

To show S has no minimal upper bound, suppose γ is a minimal upper bound and derive a contradiction.

By an earlier fact, we must have $CARD(\gamma) = 2db + I$.

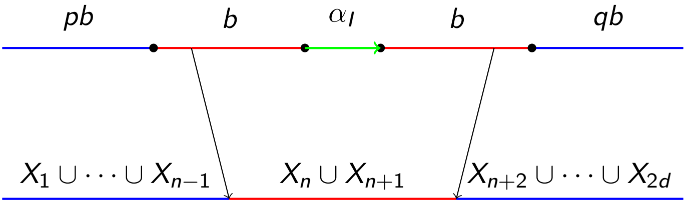
Embed $2db$ into γ. $CARD(\gamma) = 2db + l$.



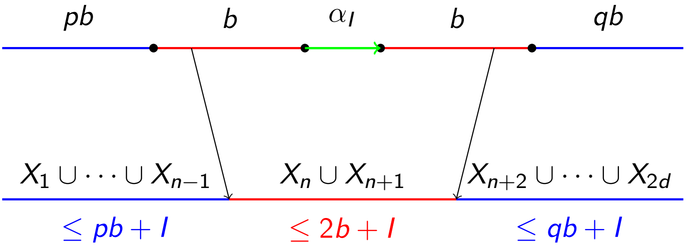
This divides γ into intervals. $b \le CARD(X_n) \le b + l$.

We will show $CARD(X_n \cup X_{n+1}) = 2b + l$ for $0 < n < 2d$.

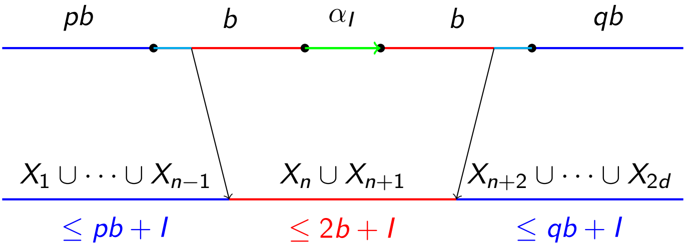
Embed $nb + \alpha_I + (2d - n)b$ into γ.



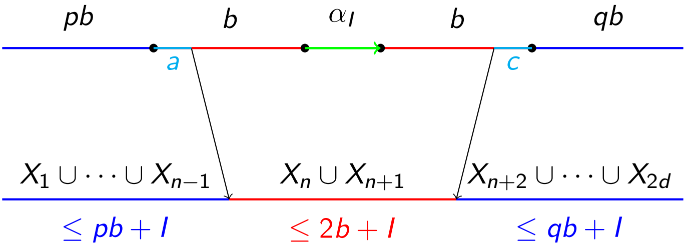
Embed $nb + \alpha_I + (2d - n)b$ into γ.



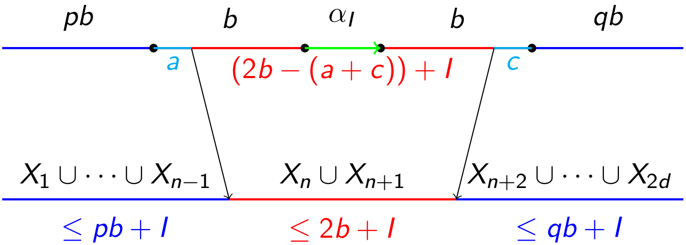
Embed $nb + \alpha_I + (2d - n)b$ into γ.



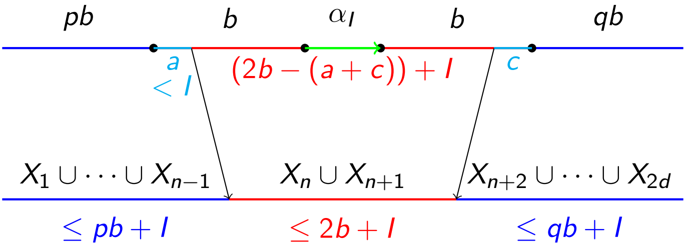
Embed $nb + \alpha_I + (2d - n)b$ into γ.



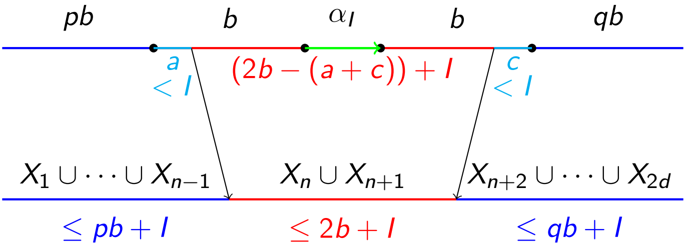
Embed $nb + \alpha_I + (2d - n)b$ into γ.



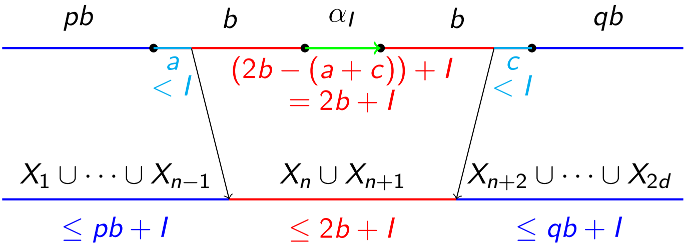
Embed $nb + \alpha_I + (2d - n)b$ into γ.



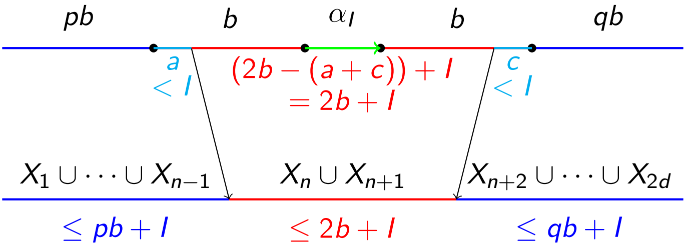
Embed $nb + \alpha_I + (2d - n)b$ into γ.



Embed $nb + \alpha_l + (2d - n)b$ into γ.



Embed $nb + \alpha_I + (2d - n)b$ into γ.



$CARD(X_n \cup X_{n+1}) = 2b + I.$

Now we have partitioned γ into d-many intervals,
$Y_k = X_{2k-1} \cup X_{2k}$ for $1 \leq k \leq d$.

$CARD(Y_k) = 2b + l > 2b + 1$.

Let Z_k be the least size-$(2b+1)$ subset of Y_k, and $Z = \bigcup_{k=1}^{d} Z_k$.

$CARD(Z) = 2db + b > 2db + l = CARD(\gamma)$.

This is a contradiction.

Question

How strong is this statement?

Every well-ordered sequence of ordinals has a minimal upper bound with respect to $\leq_w$.

References

Factorization of polynomials and Σ_1^0 induction
Stephen G. Simpson and Rick L. Smith
1986
Annals of Pure and Applied Logic

A survey of the reverse mathematics of ordinal arithmetic
Jeffry L. Hirst
2005
Reverse Mathematics 2001 ed. S. G. Simpson

THANK YOU