DMIF, Università di Udine, Italy

Provable better partial orders in reverse mathematics

Alberto Marcone

Joint work with Anton Freund, Fedor Pakhomov and Giovanni Soldà

Computability and Combinatorics 2023
May 19–21, 2023

Outline

A partial order $\mathcal{P} = (P, \leq_P)$ is a well partial order (wpo) if for every $f : \mathbb{N} \to P$ there exists $i < j$ such that $f(i) \leq_P f(j)$.

There are many equivalent characterizations of wpos:

- $\mathcal{P}$ is well founded and has no infinite antichains;
- every sequence in P has a weakly increasing subsequence;
- every nonempty subset of P has a finite set of minimal elements;
- all linear extensions of $\mathcal{P}$ are well orders.

The reverse mathematics of these equivalences has been studied in detail starting from a 2004 paper (Cholak-M-Solomon): all equivalences are provable in WKL_0+CAC.

RCA_0 proves that finite posets and well orders are wpos.

Theorem (Simpson 1988, Clote 1990)

Over RCA_0, the following are equivalent:

1. ACA_0;
2. *if $\mathcal{P}$ is a wpo then embeddability of finite strings from P is a wpo.*

Theorem (Friedman 1985)

ATR_0 *does not prove Kruskal's theorem asserting that embeddability on finite trees is a wpo.*

Theorem (Friedman-Robertson-Seymour 1987)

$\mathbf{\Pi}^1_1$-CA_0 *does not prove the Graph Minor Theorem asserting that the minor relation on finite graphs is a wpo.*

Fraïssé’s Conjecture is the statement that embeddability on countable linear orders is a wqo. We keep it distinct from Laver’s Theorem, i.e. the stronger statement that embeddability on countable linear orders is a bqo.

Theorem (Montalbán 2017)

$\mathbf{\Pi}^1_1$-CA_0 *proves Laver’s Theorem and hence Fraïssé’s Conjecture.*

Theorem (Shore 1993)

Over RCA_0, *Fraïssé’s Conjecture implies* ATR_0.

Better partial orders

The notion of better partial order (bpo) is a strengthening of wpo due to Nash-Williams (1960's).

The property of being bpo is preserved by more operations than those preserving the property of being wpo.

The general pattern is: if wpos are closed under a finitary operation, bpos are closed under its infinitary generalization.

Theorem (Pouzet, 1972)

If $\mathcal{P}$ is a poset, the following are equivalent:

1. *$\mathcal{P}$ is bpo;*
2. *the set of countable transfinite sequences of elements of P is wqo under embeddability;*
3. *the set of countable transfinite sequences of elements of P is bqo under embeddability.*

Definition of bpo

- If $X \subseteq \mathbb{N}$, we identify elements of $[X]^{<\omega} \cup [X]^{\omega}$ with the strictly increasing sequences enumerating them.
- $s \sqsubset t$ means that s is a proper initial segment of t.
- $s \subset t$ means that s is a proper subset of t (as sets).
- $B \subseteq [\mathbb{N}]^{<\omega}$ is a block if $\bigcup B$ is infinite and each $X \in [\bigcup B]^{\omega}$ admits a unique $s \sqsubset X$ with $s \in B$ (hence B is prefix-free).
- A block B such that $s \subset t$ holds for no $s, t \in B$ is a barrier.
- A $\mathcal{P}$-array is a function $f : B \to P$ on some barrier B.
- For $X \subseteq \mathbb{N}$ let $X^{-} := X \setminus \{\min X\}$.
- $s \lhd t$ means that there exists X with $s \sqsubset X$ and $t \sqsubset X^{-}$ ($\lhd$ is decidable since it only depends on $s \cup t$).
- A $\mathcal{P}$-array $f : B \to P$ is good if there are $s, t \in B$ with $s \lhd t$ and $f(s) \leq_P f(t)$. Otherwise f is bad.
- $\mathcal{P}$ is bpo if every $\mathcal{P}$-array is good.

Closure properties of bpos

Fact

RCA_0 *proves that the sum of two bpos is bpo.*
RCA_0 *proves that well orders are bpos.*

Lemma (M 2005)

ATR_0 *proves that the disjoint sum and the product of two bpos are bpos.*
Over RCA_0*, each of these two statements implies* ACA_0*.*

The notion of bpo is $\mathbf{\Pi}^1_2$-complete (M 1993) and thus "$\mathcal{P}$ is bpo" is a $\mathbf{\Pi}^1_2$ statement even when $\mathcal{P}$ is finite.

Fact

ATR_0 *proves that any finite poset is bpo.*

Let n and $\overline{n}$ be the chain and the antichain with n elements.

Lemma (M 2005)

RCA_0 *proves that* $\overline{2}$ *is bpo.*
For any $n \geq 3$, RCA_0 *proves that if* $\overline{3}$ *is bpo then any poset with* n *elements is bpo.*

Question (M 2005, Montalbán 2011)

What is the strength of " $\overline{3}$ *is bpo"?*

3 is much larger than 2

Theorem (Freund 2022)

Over RCA_0*, "* $\overline{3}$ *is bpo" implies* ACA_0^+*.*
RCA_0 *proves that if* $\overline{3}$ *is bpo then any finite poset is bpo.*

Theorem (Freund-M 2022)

ACA_0^+ *does not prove that* $\overline{3}$ *is bpo.*

Thus currently we know

$$\mathsf{ACA}_0^+ < \overline{3} \text{ is bpo} \leq \mathsf{ATR}_0$$

Therefore each of the statements "the disjoint sum of two bpos is a bpo" and "the product of two bpos is a bpo" implies ACA_0^+ and is not provable in ACA_0^+.

A natural question

In the wake of Freund's breakthrough result it is natural to ask:

If a theory T does not prove that $\overline{3}$ is bpo then what (finite) posets does T prove to be bpo?

We start with the other posets of cardinality three.

Since RCA_0 proves that well orders are bpo and that the sum of two bpos is bpo, RCA_0 proves that 3, $1 + \overline{2}$ and $\overline{2} + 1$ are bpos.

We still need to consider $1 \oplus 2$.

Theorem (Freund-M-Pakhomov-Soldà 2023)

RCA_0 *proves that* $1 \oplus 2$ *is bpo if and only if* $\overline{3}$ *is bpo.*
Thus ACA_0^+ *does not prove that* $1 \oplus 2$ *is bpo.*

Sums of antichains

Proposition (Freund-M-Pakhomov-Soldà 2023)

RCA_0 *proves that a poset* $\mathcal{P}$ *does not contain* $1 \oplus 2$ *as a suborder iff it is a linear sum of antichains (i.e.* $\mathcal{P} = \sum_{i \in \mathcal{I}} \mathcal{A}_i$ *where* $\mathcal{I}$ *is a chain and each* $\mathcal{A}_i$ *is an antichain).*

Corollary (Freund-M-Pakhomov-Soldà 2023)

RCA_0 *proves that a poset does not contain* $1 \oplus 2$ *and* $\overline{3}$ *as suborders iff it is a linear sum of antichains of size at most two.*

Proposition (Freund-M-Pakhomov-Soldà 2023)

RCA_0 *proves that a well-ordered sum of bpos (i.e.* $\sum_{i \in \mathcal{I}} \mathcal{P}_i$ *where* $\mathcal{I}$ *is a well-order and each* $\mathcal{P}_i$ *is bpo) is a bpo.*

Provable finite bpos

Theorem (Freund-M-Pakhomov-Soldà 2023)

Suppose T *extends* RCA_0 *and does not prove that* $\overline{3}$ *is bpo.*
For any finite poset $\mathcal{P}$*, the following are equivalent:*

1. T *proves that* $\mathcal{P}$ *is bpo;*
2. $\mathcal{P}$ *does not contain* $1 \oplus 2$ *and* $\overline{3}$ *as suborders;*
3. $\mathcal{P}$ *is a linear sum of antichains of size at most two.*

Provable bpos

Theorem (Freund-M-Pakhomov-Soldà 2023)

For any countable poset $\mathcal{P}$, the following are equivalent:

1. *there is a computable presentation of some $\mathcal{P}_0 \cong \mathcal{P}$ such that ACA_0 proves that $\mathcal{P}_0$ is bpo;*
2. *$\mathcal{P}$ is isomorphic to a computably enumerable suborder of $\overline{2} \cdot \gamma$ for some $\gamma < \varepsilon_0$ (ε_0 is represented by a standard notation system).*

A similar result holds for any T that does not prove that $\overline{3}$ is bpo: substitute ε_0 with the proof-theoretic ordinal of T.

Condition 1 is similar to the definition of provable well order.

Condition 2 is more complex than in the case of linear orders because in the linear case computability is automatic and the results on provable well orders avoid reference to a standard notation system.

A $\mathcal{P}$-array $f : B \to P$ (even when B is a block) can be identified with a continuous function $F : [\bigcup B]^\omega \to P$.
Vice versa, for any continuous $F : [V]^\omega \to P$ with $V \in [\mathbb{N}]^\omega$ we can find a $\mathcal{P}$-array $f : B \to P$ on a block B with $\bigcup B = V$ inducing it.

We abuse terminology, and call such an F a $\mathcal{P}$-array.

Such an F is bad if $F(X) \not\leq_P F(X^-)$ for all $X \in [V]^\omega$.

WKL_0 proves that $\mathcal{P}$ is bpo precisely when there is no bad $\mathcal{P}$-array in this new sense.

The minimal bad array lemma

Definition (RCA_0)

A partial ranking of a poset $\mathcal{P}$ is a well-founded partial order $\leq'$ on P such that $p \leq' q$ implies $p \leq_P q$.
If $F : [V]^\omega \to P$ and $G : [W]^\omega \to P$ are $\mathcal{P}$-arrays with $V \subseteq W$:

- $F \leq' G$ iff $F(X) \leq' G(X)$ for all $X \in [V]^\omega$;
- $F <' G$ iff $F(X) <' G(X)$ for all $X \in [V]^\omega$.

A $\leq'$-minimal bad $\mathcal{P}$-array is a bad $\mathcal{P}$-array G with no bad $\mathcal{P}$-array $F <' G$.

Every well-founded poset is a partial ranking of itself.

MBA If $\leq'$ is a partial ranking of $\mathcal{P}$, for any bad $\mathcal{P}$-array H there exists a $\leq'$-minimal bad $\mathcal{P}$-array $G \leq' H$.

MBA^- If $\mathcal{P}$ is well-founded and not bpo, there exists a $\leq_P$-minimal bad $\mathcal{P}$-array.

Minimal bad arrays when there are few bpos

Theorem (Freund-Pakhomov-Soldà 2023)

Over ATR_0*, the following are equivalent:*

1. $\mathbf{\Pi}^1_2$-CA_0;
2. MBA;
3. MBA^-.

Theorem (Freund-M-Pakhomov-Soldà 2023)

RCA_0 *proves that if* $\overline{3}$ *is not bpo then* MBA^- *holds.*
Thus MBA^- *does not imply* ATR_0 *over* ACA_0^+.

Theorem (Freund-M-Pakhomov-Soldà 2023)

RCA_0 *proves that* MBA *implies* ACA_0.

If $\mathcal{P}$ is a poset we define the poset $H_f(\mathcal{P}) = (H_f(P), \leq_{H(\mathcal{P})})$ of the hereditarily finite sets with urelements from P as follows:

- $P \subseteq H_f(P)$,
- if $a \subseteq H_f(P)$ is finite then $a \in H_f(P)$.

$$
\begin{aligned}
p \leq_{H(\mathcal{P})} q \quad &\Leftrightarrow \quad p \leq_P q, \\
p \leq_{H(\mathcal{P})} a \quad &\Leftrightarrow \quad \exists y \in a\, p \leq_{H(\mathcal{P})} y, \\
a \leq_{H(\mathcal{P})} p \quad &\Leftrightarrow \quad \forall x \in a\, x \leq_{H(\mathcal{P})} p, \\
a \leq_{H(\mathcal{P})} b \quad &\Leftrightarrow \quad \forall x \in a\, \exists y \in b\, x \leq_{H(\mathcal{P})} y.
\end{aligned}
$$

The elements of $H_f(P)$ can be represented by finite trees with leaf labels from P. This allows to define $H_f(\mathcal{P})$ in RCA_0.

Theorem (Freund 2022)

RCA_0 *proves that if* $\mathcal{P}$ *is bpo then* $H_f(\mathcal{P})$ *is bpo.*

From $1 \oplus 2$ to $\overline{3}$

We use $H_f(1 \oplus 2)$ to show that if $1 \oplus 2$ is bpo then $\overline{3}$ is bpo.

$H_f(1 \oplus 2)$ contains two interlocked copies of the natural numbers.
Suppose $1 \oplus 2 = \{\star\} \cup \{0, 1\}$ with $0 < 1$.

For $n \in \mathbb{N}$ define $\dot{n}, \ddot{n} \in H_f(1 \oplus 2)$ recursively by
$\dot{n} = \{\star, 0\} \cup \{\, \dot{m} \mid m < n \,\}$ $\qquad$ $\ddot{n} = \{\star, 1\} \cup \{\, \ddot{m} \mid m < n \,\}$

For $m, n \in \mathbb{N}$ we have

$$m \leq n \Leftrightarrow \dot{m} \leq_{H(1\oplus2)} \dot{n} \Leftrightarrow \ddot{m} \leq_{H(1\oplus2)} \ddot{n} \Leftrightarrow \dot{m} \leq_{H(1\oplus2)} \ddot{n}$$

while $\ddot{m} \nleq_{H(1\oplus2)} \dot{n}$.

To see that $\overline{3}$ is a suborder of $H_f(1 \oplus 2)$, consider

$$\{\ddot{0}, \dot{5}\}, \quad \{\ddot{1}, \dot{4}\}, \quad \{\ddot{2}, \dot{3}\}$$

Thank you for your attention!