Reverse Mathematics: A Global View

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May 19, 2023

Introduction

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There was then a shift in focus from individual results about the order in $\mathcal{D}_T$ to an interest in, and then emphasis on, more global questions about the its structure. Other degrees of computational complexity were then similarly analyzed.

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There was then a shift in focus from individual results about the order in $\mathcal{D}_T$ to an interest in, and then emphasis on, more global questions about the its structure. Other degrees of computational complexity were then similarly analyzed.

So it occurred to me that perhaps one should view the reverse mathematics zoo and it ordering as we do for degree structures and see what one could say about it.

Move to a Global View: Relative Provability

The elements of the underlying structure were to be theories of reverse mathematics, i.e. sets S of sentences in the language of second order arithmetic containing RCA_0. The ordering $S \leq_P T$ was to be given by $T \vdash S$, i.e. $(\forall \varphi \in S)(T \vdash \varphi)$.

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As usual our basic structure was to consist of the equivalence classes $\mathbf{s}$ of this ordering, i.e. the degrees of provability $\mathbf{s} = \{T | T \vdash S \ \& \ S \vdash T\}$ with the induced ordering. The basic structure $\mathcal{D}_P$ would then be these classes with the induced provability ordering.

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Much to my surprise there was quite a lot that could be said using mostly only basic notions and techniques as would be found in many first courses in logic: the deduction, compactness and completeness theorems; variations on incompleteness theorems and essential undecidability; as well as a couple of other classical techniques such as quantifier elimination and back and forth constructions.

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Similar Structures

I asked some proof theorists who did not know of such studies and suggested only a similar ordering, $S \leq_I T$ (S is interpretable in T) introduced by Tarski to study the ordering of consistency strength for arbitrary theories. The abstract structure of degrees of interpretability of extensions of PA (or weaker theories) has been studied by Per Lindstrom and others. In this setting, $S \leq_I T \Leftrightarrow T \vdash \varphi$ for every Π^0_1 sentence φ provable in S.

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Neither of these dealt with the provability of arbitrary sets of sentences of arithmetic as a degree structure on theories.

Tarski's Calculus of Systems

This is still work in progress but after proving most of the results I will mention, I learned from a JSL paper by Blok and Pigozzi (1988) on Tarski's work on general metamathematics that even the basic study of the ordering of theories under provability in quite general settings was also initiated by Tarski under the name of the calculus of systems in a number of papers in the 1930s.

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In any case, the analysis of this structure for reverse mathematics paints a picture of $\mathcal{D}_P$ quite different from that now known for the Turing degrees $\mathcal{D}_T$, the r.e. degrees $\mathcal{R}_T$ and many other degree structures.

Degree Structures for Relative Provability of Theories

Along with $\mathcal{D}_P$ we want to study two natural substructures: $\mathcal{F}_P$ consists of the degrees of theories finitely axiomatizable over RCA_0. $\mathcal{R}_P$ consists of the degrees of theories recursively (or equivalently, recursively enumerably) axiomatizable over RCA_0.

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As we proceed, we contrast the results for $\leq_P$ with those for $\leq_T$ on $\mathcal{D}_T$ and $\mathcal{R}_T$.

$\mathcal{D}_T$ is an upper semi-lattice with 0 of size 2^ω. $\mathcal{R}_T$ is a countable upper semi-lattice with 0. Neither is a lattice, neither is distributive neither is complete.

General Description of P-degree Structures

For $\mathcal{D}_P$ we define join and infimum based on operations on theories. They are well defined on the P-degrees and also work in $\mathcal{R}_P$ and $\mathcal{F}_P$. The three structures also all share the same least and greatest elements.

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For S any one of our theories, we let $\bar{S} = \{\varphi | S \vdash \varphi\}$. We then define operations $\vee$ and $\wedge$ and $S \vee T = \overline{(S \cup T)}$; $S \wedge T = \overline{S} \cap \overline{T}$. We let $\mathbf{0}$ and $\mathbf{1}$, respectively, be the degrees of RCA_0 and $\{\psi | \psi \in \mathcal{L}\} \equiv_P \{\varphi, \neg\varphi\}$ for any sentence φ of $\mathcal{L}$.

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Note that $S \leq_P T \Leftrightarrow \bar{S} \subseteq \bar{T}$ and so $S \equiv_P T \Leftrightarrow \bar{S} = \bar{T}$. Thus each degree $\mathbf{s}$ has a canonical representative S which is closed under deductions. Clearly $\leq_P$ on theories induces a partial order $\leq_P$ on $\mathcal{D}_P$ for which $\mathbf{0}$ and $\mathbf{1}$ are the least and greatest elements while $\vee$ and $\wedge$ induce join and inf operations on $\mathcal{D}_P$. $\mathcal{R}_P$ and $\mathcal{F}_P$ both contain $\mathbf{0}$ and $\mathbf{1}$ and are closed under $\wedge$ and $\vee$.

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Basic Structure of P-degrees

Theorem

$\mathcal{D}_P$ is a complete distributive lattice of size 2^ω with 0 and 1. $\mathcal{R}_P$ and $\mathcal{F}_P$ are countable (incomplete) distributive lattices with 0 and 1.

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Proof: Unravel the definitions and use predicate logic. For distributivity we show that $\mathbf{r} \vee (\mathbf{s} \wedge \mathbf{t}) = (\mathbf{r} \vee \mathbf{s}) \wedge (\mathbf{r} \vee \mathbf{t})$ by applications of predicate logic, For the more interesting containment direction say $\varphi \in (R \vee S) \wedge (R \wedge T)$. So $\exists \rho_1, \rho_2 \in R \exists \sigma \in S \exists \tau \in T(\rho_1 \& \sigma \vdash \varphi,\ \rho_2 \& \tau \vdash \varphi)$. Let $\rho = \rho_1 \& \rho_2 \in R$. So $\rho \& \sigma \vdash \varphi$ and $\rho \& \tau \vdash \varphi$ i.e. $\rho \& (\sigma \textit{ or } \tau) \vdash \varphi$. As $\rho \& (\sigma \textit{ or } \tau) \in R \vee (S \wedge T)$ so is φ as required.

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For $\mathcal{D}_P$ the infinitary join and infimum are equivalent to a join or infimum of a countable subset by the compactness theorem and the countability of $\mathcal{L}$. For subsets of $\mathcal{R}_p$ or $\mathcal{F}_P$ the infinitary operations inside $\mathcal{D}_P$ may not produce an element of $\mathcal{R}_P$ or $\mathcal{F}_P$. If they do not, the set has no sup or inf in $\mathcal{R}_P$ or $\mathcal{F}_P$.

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We give more information about density in $\mathcal{R}_P$ and $\mathcal{D}_P$ by investigating the exceptions to density, i.e. the possible pairs of degrees $\mathbf{s}$ and $\mathbf{t}$ such that $\mathbf{t}$ is a *minimal cover* of $\mathbf{s}$, i.e. $\mathbf{s} < \mathbf{t}$ and there is no $\mathbf{r}$, $\mathbf{s} < \mathbf{r} < \mathbf{t}$.

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In $\mathcal{D}_T$ every degree has continuum many minimal covers. Some degrees such as all the r.e. degrees and all the $\mathbf{0}^{(n)}$ are not minimal covers. Every degree above $0^{(\omega)}$ is a minimal cover. $\mathcal{R}_T$ is dense so no minimal covers.

Minimal Covers: Some Facts

Theorem

In $\mathcal{D}_P$ *every* $\mathbf{s} \neq \mathbf{0}$ *is a minimal cover. Indeed, for any* $\varphi \in S - RCA_0$ *there is an* R *such that* $\overline{R \cup \{\varphi\}} = S$ *which is a minimal cover of* R*.*

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No $\varphi \in \mathcal{F}_P$ has a minimal cover in $\mathcal{D}_P$, $\mathcal{R}_P$ or $\mathcal{F}_P$.

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Theorem

S has a complement in $\mathcal{D}_P$ iff it has one in $\mathcal{R}_P$ (and they are then the same element) iff it is finitely axiomatizable over RCA_0, i.e. $S \in \mathcal{F}_P$. (compactness)

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The class of degrees of theories finitely axiomatizable over RCA_0 are definable in $\mathcal{D}_P$ and $\mathcal{R}_P$.

Algebraic and Heyting Latices; Pseudocomplements

Definition

Let $\mathcal{L}$ be a lattice with 0. If $a \in \mathcal{L}$ then $a^* \in \mathcal{L}$ is a *pseudocomplement of a* if $a \wedge a^* = 0$ and for any $x \in \mathcal{L}$ such that $a \wedge x = 0$, $x \leq a^*$. If every element of $\mathcal{L}$ has a pseudocomplement, $\mathcal{L}$ is *pseudocomplemented*. $\mathcal{L}$ is *relatively pseudocomplemented* (a *Heyting algebra*) if for every $a, b \in \mathcal{L}$ there is a d such that $\forall x((a \wedge x) \leq b \Leftrightarrow x \leq d)$. The *compact elements* of $\mathcal{L}$ are those a such for any $X \subseteq \mathcal{L}$ with $a \leq \vee X$ there is a finite $\hat{X} \subseteq X$ such that $a \leq \vee \hat{X}$. $\mathcal{L}$ is *algebraic* if every element is the join of the compact elements below it.

Algebraic and Heyting Latices; Pseudocomplements

Definition

Let $\mathcal{L}$ be a lattice with 0. If $a \in \mathcal{L}$ then $a^* \in \mathcal{L}$ is a *pseudocomplement of a* if $a \wedge a^* = 0$ and for any $x \in \mathcal{L}$ such that $a \wedge x = 0$, $x \leq a^*$. If every element of $\mathcal{L}$ has a pseudocomplement, $\mathcal{L}$ is *pseudocomplemented*. $\mathcal{L}$ is *relatively pseudocomplemented* (a *Heyting algebra*) if for every $a, b \in \mathcal{L}$ there is a d such that $\forall x((a \wedge x) \leq b \Leftrightarrow x \leq d)$. The *compact elements* of $\mathcal{L}$ are those a such for any $X \subseteq \mathcal{L}$ with $a \leq \vee X$ there is a finite $\hat{X} \subseteq X$ such that $a \leq \vee \hat{X}$. $\mathcal{L}$ is *algebraic* if every element is the join of the compact elements below it.

Theorem

$\mathcal{D}_P$ is a complete algebraic lattice and relatively pseudocomplemented (a Heyting algebra). $\mathcal{R}_P$ is an incomplete algebraic lattice. For each of them the compact elements are those in $\mathcal{F}_P$ and the pseudocomplement of S relative to T in $\mathcal{D}_P$ is $\vee\{\overline{\{\varphi\}} | \overline{\{\varphi\}} \wedge S \leq T\}$.

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Homogeneity generally fails: If **a** and **b** are degrees with a noticeable difference between them (in terms of a few Turing jumps of each) then the cones $\mathcal{D}_T^{\mathbf{a}}$ (the T-degrees above **a**) and $\mathcal{D}_T^{\mathbf{b}}$ are not isomorphic. No two distinct cones are known to be isomorphic.

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The theory of $\mathcal{R}_T$ is recursively isomorphic to the true first order theory of arithmetic. Several interesting classes of r.e. degrees are naturally definable and many others have definitions. It is not known if $\mathcal{R}_T$ has a nontrivial automorphism or a definable degree.

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Generating Sets, Automorphisms, Definability: P-Degrees

Theorem

$\mathcal{F}_P$ generates both $\mathcal{D}_P$ and $\mathcal{R}_P$ under infinitary joins and so is an automorphism basis for each of them. (All such joins exist in $\mathcal{D}_P$. For $\mathcal{R}_P$ we only need the ones which do exist there.)

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Theorem

$\mathcal{F}_P$ and $\mathcal{D}_P$ have exactly 2^ω many automorphisms. If $\mathbf{s}, \mathbf{t} \in \mathcal{F}_\mathbf{P}$ and neither is $\mathbf{0}$ or $\mathbf{1}$, there is an automorphism of $\mathcal{F}_P$ and so of $\mathcal{D}_P$ taking $\mathbf{s}$ to $\mathbf{t}$. If $\mathbf{s}$ and $\mathbf{t}$ are coatoms, there is an automorphism of $\mathcal{D}_P$ taking $\mathbf{s}$ to $\mathbf{t}$.

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Corollary

There are no definable P-degrees other than $\mathbf{0}$ and $\mathbf{1}$.

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Assume $\mathcal{S}$ is a countable subset of $\mathcal{D}_P$ not on the list above. We want to construct an automorphism of $\mathcal{F}_P$ which extends to one of $\mathcal{D}_P$ that does not fix $\mathcal{S}$. Wlog we may assume that $0, 1 \in \mathcal{S}$. If $\mathcal{S}$ splits $\mathcal{F}_P - \{0,1\}$ then choose $\varphi, \psi \in \mathcal{F}_P$ with $\varphi \in \mathcal{S}$ and $\psi \notin \mathcal{S}$. Any automorphism of $\mathcal{F}_P$ $\varphi \mapsto \psi$ is as required.

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Otherwise, $\mathcal{F}_P \subsetneq \mathcal{S}$ or $\mathcal{F}_P \cap \mathcal{S} = \{0, 1\}$. In either case we may choose an $S \in \mathcal{S}$ with $S \notin \mathcal{F}_P$. List the $S_i \in \mathcal{S}$ and build an automorphism of $\mathcal{F}_P$ in stages n that takes one finite Boolean subalgebra φ_i to another ψ_i, $i < k_n$.

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In addition to the usual moves to make the map an automorphism of $\mathcal{F}_P$, at some stages we want to extend the finite automorphism constructed so far to guarantee that at the end its extension to $\mathcal{D}_P$ does not take S to S_m where m is least such that we have not yet guaranteed that the automorphism does not take S to S_m.

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The conditions for extending the automorphism require that the new item θ chosen on the φ side is matched with something γ satisfying the same relations on the ψ side relative to the already defined map. The fact that $S \notin \mathcal{F}_P$ means that one can fix the constraints satisfied by S with respect to the φ_i without fixing S. One then can rule out any S_m not satisfying the same constraints with respect to the ψ_i. Moreover, one can rule out any $S_m \in \mathcal{F}_P$. One can then use the algebraic structure to find a $\gamma \notin S_m$ satisfying the constraints with respect to the ψ_i that are the same as ones satisfied by θ with respect to the φ_i. One can then extend the automorphism to the next pair of finite Boolean algebras so that θ is sent to γ.

Homogeneity

Thanks to Leo Harrington for a correction to an earlier version of the following.

Theorem

The cones $\mathcal{D}_P^{\mathbf{s}}$ for $\mathbf{s} < \mathbf{1}$ correspond to the countable Boolean algebras with the set of their ideals (identify principal ideals with their generator). If $S \vee \varphi$ is not a complete theory for any φ, then the atomless Boolean algebra $\mathcal{F}_P \cong \mathcal{F}_P^s$, the degrees of theories finitely axiomatizable over S. So $\mathcal{D}_P^{\mathbf{s}}$ is the the closure of $\mathcal{F}_P^s$ under infinite joins and $\mathcal{D}_P^{\mathbf{s}} \cong \mathcal{D}_P$.

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Thus strengthening the base theory, RCA_0, to any with no complete finite extensions gives the same structures for $\mathcal{F}_P$ and $\mathcal{D}_P$. These include any recursively axiomatized extension of RCA_0. Thus for any such base theory S, $\mathcal{D}_P^{\mathbf{s}} \cong \mathcal{D}_P$ and so the zoo over S is identical to the one for RCA_0.

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If we instead look at weaker theories of second order arithmetic, our analysis works as long as it is strong enough to be subject to the same incompleteness phenomena as RCA_0.

Questions

$\mathcal{R}_P$: The most interesting class of questions concern the structure of $\mathcal{R}_P$, the recursively axiomatizable theories, i.e. the ones with r.e. representatives. In particular, what can one say about the automorphisms of $\mathcal{R}_P$? We have almost no information. The problem, of course, is that the back and forth constructions of automorphisms of $\mathcal{F}_P$ are are not recursive. So the question is if there is some way to preserve recursive enumerability.

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Orbits: The complete theories and the finitely axiomatized theories each form an orbit (and a definable set). Are there other definable orbits? For example, is it true that for every finite Boolean algebra B (or even for any other than $\{0.1\}$) if $\mathcal{D}_P^{\mathbf{s}} \cong \mathcal{D}_P^{\mathbf{t}} \cong B$, then there is an automorphism of $\mathcal{D}_P$ taking **s** to **t**.

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Thanks.