Weihrauch degrees above arithmetical transfinite recursion

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Summary

I will talk about

- studies on the complexity of mathematical theorems/problems.
  - Reverse Mathematics: Which axiom is needed to prove?
  - Weihrauch Degrees: How difficult is it to construct solutions?
- Typically, consider the complexity of arithmetical statements above the level of $\mathbf{ATR}_0$ / hyperarithmeticity from the viewpoint of Weihrauch degrees.

Systems of second-order arithmetic

$\mathcal{L}_2$-systems used in Reverse Math

$$\mathbf{RCA}_0 = \mathbf{PA}^- + \mathbf{I}\Sigma^0_1 + \Delta^0_1\text{-}\mathbf{CA},$$
$$\mathbf{ACA}_0 = \mathbf{RCA}_0 + \forall X \exists Y (Y = X' = \mathsf{Jump}(X)),$$
$$\mathbf{ATR}_0 = \mathbf{RCA}_0 + \forall W, X (\mathrm{WO}(W) \to \exists Y \, \mathrm{Hier}(X, W, Y)).$$

Here $\mathrm{Hier}(X, L, \langle Y_i \rangle_i)$ denotes the following formula:

$$Y_{\min L} = X \wedge (\forall i \neq \min L)(Y_i = \mathsf{Jump}(\langle Y_j \rangle_{j <_L i})).$$

Intuition of Hier

Y is the W-time iteration of Turing jump of X.

Y is a (pseudo-)jump hierarchy for (L, X) if $\mathrm{Hier}(X, L, Y)$ holds.

Weihrauch degrees

Definition

Weihrauch problem:

a partial function $\mathsf{P} :\subseteq \mathcal{P}(\omega) \to \mathcal{P}(\mathcal{P}(\omega))$.

- $X \in \mathrm{dom}(\mathsf{P})$ is called *an input for* X.
- For $X \in \mathrm{dom}(\mathsf{P})$, Y *is an output of* $\mathsf{P}(X)$ if $Y \in \mathsf{P}(X)$.

P, Q : problems.

Definition (Weihrauch Reduction)

P is Weihrauch reducible to Q (denoted by $\mathsf{P} \leq_W \mathsf{Q}$) if there are computable functionals Φ, Ψ such that

$(\forall X \in \mathrm{dom}\,\mathsf{P})(\Phi(X) \downarrow \in \mathrm{dom}\,\mathsf{Q})$,

$(\forall X \in \mathrm{dom}\,\mathsf{P})(\forall Y)((\Phi(X), Y) \in Q) \to (X, \Psi(X, Y) \downarrow) \in \mathsf{P})$.

$\mathsf{P} \leq_W \mathsf{Q}$ via Φ, Ψ shows the following condition:

$$\begin{array}{cccc}
\text{Input for } \mathsf{P} & X & \Psi(X \oplus Y) & \text{Output of } \mathsf{P}(X) \\
 & \downarrow & \uparrow & \\
\text{Input for } \mathsf{Q} & \Phi(X) \longrightarrow & Y & \text{Output of } \mathsf{Q}(\Phi(X))
\end{array}$$

Definition (Weihrauch degrees)

The degree structure of Weihrauch Problems induced by $\leq_W$ is called Weihrauch hierarchy.

$\mathcal{L}_2$-statements and Weihrauch problems

Let T be an $\mathcal{L}_2$-statement of the form $\forall X(\theta(X) \to \exists Y \eta(X, Y))$.

e.g. if X is an ill-founded linear order on $\mathbb{N}$ then Y is an infinite X-descending sequence.

Put $\mathsf{P}(\theta, \eta) = \{(X, Y) : (\omega, \mathcal{P}(\omega)) \models \theta(X) \land \eta(X, Y)\}$.

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We focus on Weihrauch problems described by $\mathcal{L}_2$-formulas.

- $\mathsf{P} = \mathsf{P}(\theta, \eta)$ is said to be an arithmetical problem if both of θ and η are arithmetical.
- $\mathsf{P} = \mathsf{P}(\theta, \eta)$ is said to be a (Γ, Δ)-problem if $\theta \in \Gamma$ and $\eta \in \Delta$, where $\Gamma, \Delta \in \{\Sigma^1_1, \Pi^1_1 \dots\}$
- Conversely, if $\mathsf{P} = \mathsf{P}(\theta, \eta)$, we may write $\mathcal{L}_2(\mathsf{P})$ for the $\mathcal{L}_2$-statement $\forall X(\theta(X) \to \exists Y \eta(X, Y))$.

Definition (ω-model reduction)

Let P, Q be problems.

- P is ω-model reducible to Q ($\mathsf{P} \leq_\omega \mathsf{Q}$) if for any $S \subseteq \mathcal{P}(\omega)$, $(\omega, S) \models \mathbf{RCA}_0 + \mathcal{L}_2(\mathsf{Q})$ implies $(\omega, S) \models \mathcal{L}_2(\mathsf{P})$.
- P is arithmetically ω-model reducible to Q ($\mathsf{P} \leq_\omega^a \mathsf{Q}$) if for any $S \subseteq \mathcal{P}(\omega)$, $(\omega, S) \models \mathbf{ACA}_0 + \mathcal{L}_2(\mathsf{Q})$ implies $(\omega, S) \models \mathcal{L}_2(\mathsf{P})$.

- We may easily see that $\leq_W \subseteq \leq_\omega \subseteq \leq_\omega^a$.
- In other words, the study of the Weihrauch degrees is a refinement of the study of ω-models of second-order arithmetic.

1. Second-order arithmetic and Weihrauch degrees
2. Around arithmetical transfinite recursion
3. Above arithmetical transfinite recursion

Definition (monotone operator)

- An operator $\Gamma : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is said to be *monotone* if $X \subseteq Y \Rightarrow \Gamma(X) \subseteq \Gamma(Y)$.
- $\Gamma : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is said to be *arithmetical* if there is an arithmetical formula $\varphi(n, X)$ possibly with parameters from $\mathcal{P}(\omega)$ such that $\Gamma_\varphi(X) = \{n : \varphi(n, X)\}$.

Theorem (FP, weak form of the Knaster-Tarski theorem)

FP: any monotone operator $\Gamma : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ has a fixed point.

Question:

What is the strength of FP for arithmetical operators?

Thoerem (Avigad, 1996)

Over $\mathbf{RCA}_0$, TFAE.

1. $\mathbf{ATR}_0$,
2. FP for arithmetical operators.
3. FP for positive Σ^0_2-operators.

(An arithmetical formula $\varphi(n, X)$ is *positive* if there is no subformula of the form $t \notin X$ in the negation normal form.)

This equivalence is proved by pseudo-hierarchy method.

Question:

How about the situation in Weihrauch degrees?

Jump

Input: Any set X

Output: The Turing jump X' of X.

ATR

Input: A well-order W and a set A.

Output: The jump hierarchy of (W, A).

ATR_2

Input: A linear order L and a set A.

Output: A jump hierarchy of (L, A) or a descending sequence of L.

$\mathsf{C}_{\omega^\omega}$

Input: An ill-founded tree $T \subseteq \omega^{<\omega}$.

Output: A path of T.

Observation

- $\mathbf{RCA}_0 + \mathcal{L}_2(\mathsf{Jump}) = \mathbf{ACA}_0$.
- $\mathbf{RCA}_0 + \mathcal{L}_2(\mathsf{ATR}) = \mathbf{RCA}_0 + \mathcal{L}_2(\mathsf{ATR}_2) = \mathbf{ATR}_0$.
- $\mathcal{L}_2(\mathsf{C}_{\omega^\omega})$ is a trivial statement.

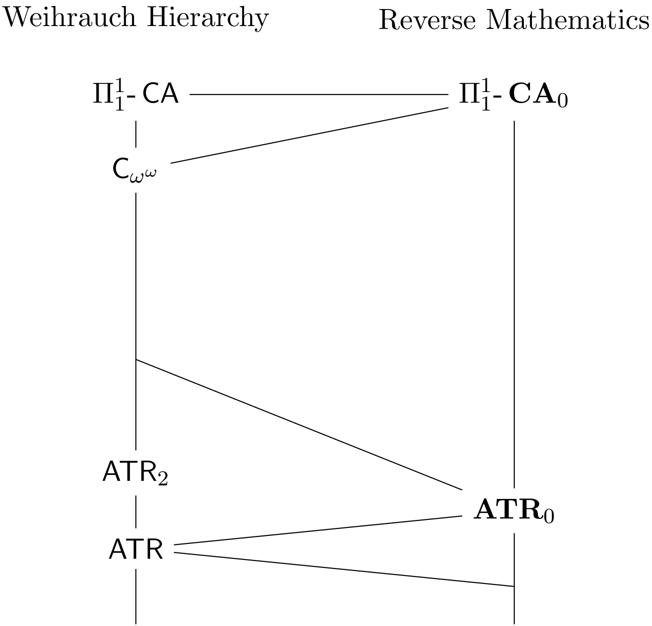
Theorem (Kihara/Marcone/Pauly, Goh)

$$\mathsf{Jump} <_W \mathsf{ATR} <_W < \mathsf{ATR}_2 <_W \mathsf{C}_{\omega^\omega} .$$

* ATR may be considered as the truth in (ω, HYP^X)-models.
* $\mathsf{C}_{\omega^\omega}$ may be considered as the truth in β-models.

Theorem (folklore?)

1. If P is $(\Sigma^1_\infty, \Sigma^1_0)$, then $\mathsf{P} \leq_W \mathsf{C}_{\omega^\omega}$.
2. If P is (Π^1_1, Σ^1_0), then $\mathsf{P} <_W \mathsf{C}_{\omega^\omega}$.
3. If P is (Σ^1_0, Σ^1_0) and $\mathsf{ATR} \leq_W \mathsf{P}$, then $\mathsf{ATR} <_W \mathsf{P} <_W \mathsf{C}_{\omega^\omega}$.


Weihrauch Hierarchy
Reverse Mathematics
Π^1_1- CA
Π^1_1- $\mathbf{CA}_0$
$\mathsf{C}_{\omega^\omega}$
ATR_2
$\mathbf{ATR}_0$
ATR

FP Σ^0_2

Input A positive Σ^0_2-formula (and its parameters) $\varphi(X)$.

Output A fixed point of Γ_φ.

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- $\mathcal{L}_2(\mathsf{FP}\,\Sigma^0_2)$ is provable from $\mathbf{ATR}_0 = \mathbf{RCA}_0 + \mathcal{L}_2(\mathsf{ATR}_2)$, but the proof cannot be converted to the reduction "$\mathsf{FP}\,\Sigma^0_2 \leq_W \mathsf{ATR}_2$".
- Indeed, the proof essentially uses the pseudo-hierarchy method.

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Theorem

- $\mathsf{ATR}_2 <_W \mathsf{ATR}_2 \times \mathsf{ATR}_2 \leq_W \mathsf{FP}\,\Sigma^0_2$,
- $\mathsf{FP}\,\Sigma^0_2$ is parallelizable, but ATR_2 is not.

Fact ($\mathbf{ACA}_0$)

There is no Σ^1_1 formula $\varphi(X)$ such that $\forall X(\varphi(X) \leftrightarrow X$ is a well-order.)

Pseudo-Hierarchy Method for FP in $\mathbf{ATR}_0$

Let $\varphi(n, X)$ be a formula and L is a well-order. Then there exists a sequence $\langle A_i \rangle_{i \in |L|}$ such that

- $A_i = \{n : \varphi(n, \bigcup_{j <_L i} A_j)\}$.

Thus there is an ill-founded linear order L and a sequence $\langle A_i \rangle_{i \in |L|}$ such that

- $\min_L\{j \in |L| \mid j >_L i\}$ exists for any $i \in |L|$,
- $A_i = \{n : \varphi(n, \bigcup_{j <_L i} A_j)\}$,
- for any $i \in |L|$ and $x \in A_i$, $\min_L\{j \in |L| \mid x \in A_j\}$ exists.

If $\langle i_s \mid s \in \mathbb{N} \rangle$ is an infinite decreasing sequence of L, then $\bigcap_s A_{i_s}$ is a fixed point.

- Formulation of psuedo-hierarchy method in [PEA d'Auriac, 2019].

  Strong enough but above $\mathsf{C}_{\omega^\omega}$.
- Some weaker formulations of the psuedo-hierarchy method may be available but they are usually equivalent to $\mathsf{C}_{\omega^\omega}$.
- In general, to apply the psuedo-hierarchy method in the setting of Weihrauch degrees, we need an ill-founded linear order L, its decreasing sequence and the witness for $\varphi(L)$.

Question:

How can we bound the psuedo-hierarchy method available within $\mathbf{ATR}_0$ in the context of Weihrauch degrees?

1. Second-order arithmetic and Weihrauch degrees
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Definition

For a problem $\mathsf{P}(\theta,\eta)$, define $\mathsf{P}^{\mathrm{rfn}}$ as follows.

Input: Any set X,

Output: A tuple $(\langle M_i\rangle_{i\in\omega}, f, g, e)$ such that

$$\langle M_i\rangle_i \text{ is closed under } \le_T,$$
$$(\forall i,j)(M_{f(i)} = M_i' \wedge M_{g(i,j)} = M_i \oplus M_j),$$
$$M_e = X,$$
$$(\forall i)(\theta(M_i) \to \exists j \eta(M_i, M_j))$$

- $\langle M_i\rangle_i$ is an ω-model of $\mathbf{ACA}_0$.
- If $\theta \in \Sigma^1_1, \eta \in \Pi^1_1$ then $\langle M_i\rangle_i$ also satisfies P.

Theorem

Let $\mathsf{P}(\theta,\eta), \mathsf{Q}(\widetilde{\theta},\widetilde{\eta})$ be problems.

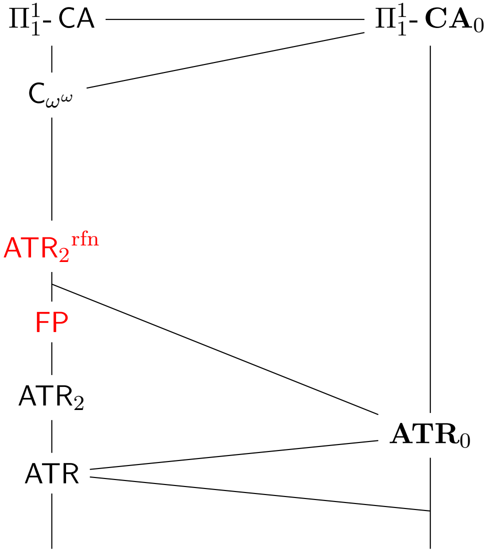
1. If $\eta \in \Sigma^1_0$ then $\mathsf{P} \leq_W \mathsf{P}^{\mathrm{rfn}}$.
2. If $\theta, \eta \in \Sigma^1_0$ then $\mathsf{P} <_W \mathsf{P}^{\mathrm{rfn}} <_W \mathsf{C}_{\omega^\omega}$.
3. If $\theta \in \Pi^1_1, \eta \in \Sigma^1_1, \widetilde{\theta}, \widetilde{\eta} \in \Sigma^1_0$ and P is provable from $\mathsf{Q} + \mathbf{ACA}_0 + \Sigma^1_\infty$-$\mathbf{IND}$, then $\mathsf{P}^{\mathrm{rfn}} \leq_W \mathsf{Q}^{\mathrm{rfn}}$.

Corollary

For any arithmetical problem $\mathsf{P}(\theta,\eta)$, if $\mathcal{L}_2(\mathsf{P})$ is provable from $\mathbf{ATR}_0 + \Sigma^1_\infty$-$\mathbf{IND}$, then $\mathsf{P}(\theta,\eta) <_W \mathsf{P}^{\mathrm{rfn}} \leq_W \mathsf{ATR}_2^{\mathrm{rfn}}$.

Especially, $\mathsf{FP}\,\Sigma^0_2 <_W (\mathsf{FP}\,\Sigma^0_2)^{\mathrm{rfn}} \equiv_W \mathsf{ATR}_2^{\mathrm{rfn}}$.

Weihrauch Hierarchy Reverse Mathematics



For the separation of P and $\mathsf{P}^{\mathrm{rfn}}$, we can show a bit more.

Theorem (ω-model incompleteness, Friedman)

Let φ be an $\mathcal{L}_2$ sentence which is true in $(\omega, \mathcal{P}(\omega))$.
Put $\mathbf{T} = \mathbf{ACA}_0 + \varphi$. Then there exists $S \subseteq \mathcal{P}(\omega)$ such that

$$(\omega, S) \models \mathbf{T} + \neg\exists \text{countable coded } \omega\text{-model of } \mathbf{T}.$$

Note that adding $\mathbf{ACA}_0$ is essential for the above theorem.

Corollary

If P is an arithmetical problem, then $\mathsf{P}^{\mathrm{rfn}} \not\leq^a_\omega \mathsf{P}$.

Question:

Are there any "natural" problems between $(\mathsf{ATR}_2)^{\mathrm{rfn}}$ and $\mathsf{C}_{\omega^\omega}$?

There is a good hierarchy of problems between $\mathsf{ATR}_2^{\mathrm{rfn}}$ and $\mathsf{C}_{\omega^\omega}$.

Definition (Σ^0_k LPP)

Input: An ill-founded tree T and its path f.

Output: A path g of T such that there is no $\Sigma_k^{T\oplus f\oplus g}$-definable path h of T lexicographically smaller than g.

Originally, this was introduced in [Towsner, 2013] in the context of reverse mathematics.

- $\mathbf{ACA}_0 + \Sigma^0_0$ LPP implies $\mathbf{ATR}_0$.
- $\Pi^1_1\mathbf{CA}_0$ implies Σ^0_k LPP, and the converse does not hold.

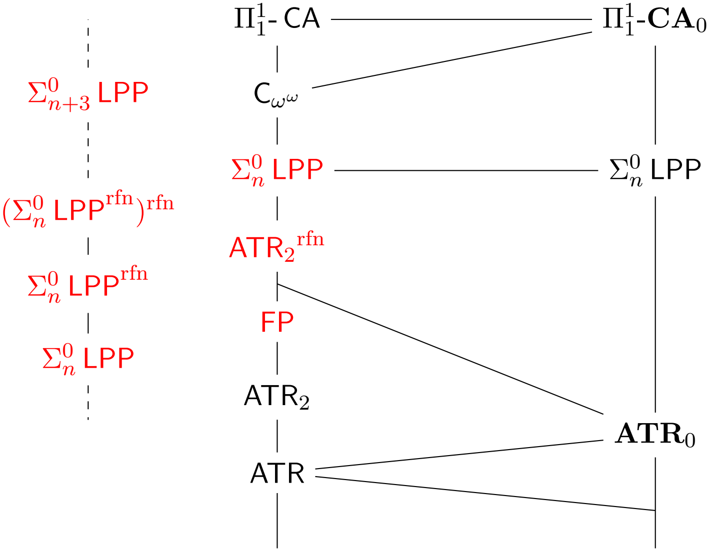
We let $(\mathsf{P})^{n+1\,\mathrm{rfn}} = ((\mathsf{P})^{n\,\mathrm{rfn}})^{\mathrm{rfn}}$.

Theorem

1. $\mathsf{ATR} \leq_W \Sigma^0_0\,\mathsf{LPP}$.
2. $\mathsf{FP}\,\Sigma^0_2 \leq_W \Sigma^0_2\,\mathsf{LPP}$.
3. $\mathsf{ATR}_2 <_W (\mathsf{ATR}_2)^{\mathrm{rfn}} <_W \Sigma^0_3\,\mathsf{LPP}$.
4. $\Sigma^0_n\,\mathsf{LPP} <_W (\Sigma^0_n\,\mathsf{LPP})^{n\,\mathrm{rfn}} <_W \Sigma^0_{n+3}\,\mathsf{LPP}$.
5. $\Sigma^0_n\,\mathsf{LPP} <_W \mathsf{C}_{\omega^\omega}$.

Moreover, for 3–5, the converse does not holds even $\leq^a_\omega$.

Weihrauch Hierarchy Reverse Mathematics



Questions:

- Does $\max_{\leq_W}\{\mathsf{P} \in (\Sigma^1_0, \Sigma^1_0) \mid \mathbf{ATR}_0 \vdash \mathcal{L}_2(\mathsf{P})\}$ exist? (Note that if 'sup' exists, then it has to be 'max'.)
- Can we improve the separation $(\Sigma^0_n\,\mathsf{LPP})^{\mathrm{rfn}} <_W \Sigma^0_{n+3}\,\mathsf{LPP}$?
- Are there more "natural" arithmetical problems above/around $\Sigma^0_n\,\mathsf{LPP}$?

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