Effective dimensions and the point to set principle for separable spaces: the Hilbert cube and the hyperspace

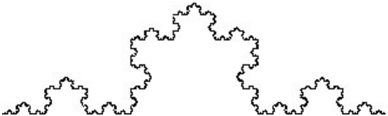
Jack Lutz, Neil Lutz, **Elvira Mayordomo**

Iowa State University, Universidad de Zaragoza

CIRM New Directions in Computability Theory, March 9th 2022

Fractal dimensions

Given a (separable) metric space, Hausdorff dimension and packing dimension generalize the usual integer dimension idea



Hausdorff definition of dimension

Let ρ be a metric on a set X.

- For $E \subseteq X$ and $\delta > 0$, a δ-cover of E is a collection $\mathcal{U}$ such that for all $U \in \mathcal{U}$, $\mathrm{diam}(U) < \delta$ and
$$E \subseteq \bigcup_{U \in \mathcal{U}} U.$$
- For $s \geq 0$,
$$H^s(E) = \lim_{\delta \to 0} \inf_{\mathcal{U} \text{ is a } \delta\text{-cover of } E} \sum_{U \in \mathcal{U}} \mathrm{diam}(U)^s$$

The Hausdorff dimension of $E \subseteq X$ is
$$\dim_{\mathrm{H}}(E) = \inf \{ s \mid H^s(E) = 0 \}.$$

Packing dimension

Let ρ be a metric on a set X.

- For $E \subseteq X$ and $\delta > 0$, a δ-packing of E is a collection $\mathcal{U}$ of disjoint open balls U with centers in E and $\mathrm{diam}(U) < \delta$.
- For $s \geq 0$,
  $$P_0^s(E) = \lim_{\delta\to 0} \sup_{\mathcal{U} \text{ is a } \delta\text{-packing of } E} \sum_{U\in\mathcal{U}} \mathrm{diam}(U)^s$$
- For $s \geq 0$,
  $$P^s(E) = \inf\left\{\sum_i P_0^s(E_i) \,\middle|\, E \subseteq \cup E_i\right\}$$

The Packing dimension of $E \subseteq X$ is
$$\dim_{\mathrm{P}}(E) = \inf\{s \,|\, P^s(E) = 0\}.$$

Effectivizing Hausdorff dimension I

Definition

An s-gale is $d : 2^{<\omega} \to [0, \infty)$ with

$$d(w) = \frac{d(w0) + d(w1)}{2^s},$$

$$S^{\infty}[d] = \left\{ x \in 2^{\omega} \,\middle|\, \limsup_n d(x \upharpoonright n) = \infty \right\}.$$

Effectivizing Hausdorff dimension I

Definition

An s-gale is $d : 2^{<\omega} \to [0, \infty)$ with

$$d(w) = \frac{d(w0) + d(w1)}{2^s},$$

$$S^\infty[d] = \left\{ x \in 2^\omega \;\middle|\; \limsup_n d(x \upharpoonright n) = \infty \right\}.$$

$$\mathrm{dim}(x) = \inf\{s \quad | \quad \text{there is a lower semicomputable } s\text{-gale } d \text{ with } x \in S^\infty[d]\}$$

Effectivizing Hausdorff dimension I

Definition

An *s-gale* is $d : 2^{<\omega} \to [0, \infty)$ with

$$d(w) = \frac{d(w0) + d(w1)}{2^s},$$

$$S^\infty[d] = \left\{ x \in 2^\omega \,\middle|\, \limsup_n d(x \upharpoonright n) = \infty \right\}.$$

$$\begin{aligned}\dim(x) = \inf\{s \quad | \quad & \text{there is a lower semicomputable} \\ & s\text{-gale } d \text{ with } x \in S^\infty[d]\}\end{aligned}$$

$$\dim(A) = \sup_{x \in A} \dim(x)$$

Effectivizing Hausdorff dimension I

Definition

An s-gale is $d : 2^{<\omega} \to [0, \infty)$ with

$$d(w) = \frac{d(w0) + d(w1)}{2^s},$$

$$S^\infty[d] = \left\{ x \in 2^\omega \,\middle|\, \limsup_n d(x \restriction n) = \infty \right\}.$$

$$\dim(x) = \inf\{s \mid \text{there is a lower semicomputable } s\text{-gale } d \text{ with } x \in S^\infty[d]\}$$

$$\dim(A) = \sup_{x \in A} \dim(x)$$

Effectivizing Hausdorff dimension II

From a compression/decompression definition:

- Fix U a UTM. Let $w \in 2^{<\omega}$, $x \in 2^{\omega}$, $\delta > 0$

$$\mathrm{K}(w) = \min\{|y| \mid U(y) = w\}$$

$$\mathrm{K}_{\delta}(x) = \inf\{\mathrm{K}(q) \mid q \in \mathbb{Q}, |x - q| < \delta\}$$

Effectivizing Hausdorff dimension II

From a compression/decompression definition:

- Fix U a UTM. Let $w \in 2^{<\omega}$, $x \in 2^{\omega}$, $\delta > 0$

$$\mathrm{K}(w) = \min \{ |y| \, | \, U(y) = w \}$$

$$\mathrm{K}_{\delta}(x) = \inf \{ \mathrm{K}(q) \, | \, q \in \mathbb{Q}, |x - q| < \delta \}$$

$$\dim(x) = \liminf_{\delta \to 0^+} \frac{\mathrm{K}_{\delta}(x)}{\log(1/\delta)}.$$

Effectivizing Hausdorff dimension II

From a compression/decompression definition:

- Fix U a UTM. Let $w \in 2^{<\omega}$, $x \in 2^{\omega}$, $\delta > 0$

$$\mathrm{K}(w) = \min\{|y| \,|\, U(y) = w\}$$

$$\mathrm{K}_\delta(x) = \inf\{\mathrm{K}(q) \,|\, q \in \mathbb{Q}, |x - q| < \delta\}$$

$$\dim(x) = \liminf_{\delta \to 0^+} \frac{\mathrm{K}_\delta(x)}{\log(1/\delta)}.$$

$$\dim(A) = \sup_{x \in A} \dim(x)$$

Effectivizing Hausdorff dimension II

From a compression/decompression definition:

- Fix U a UTM. Let $w \in 2^{<\omega}$, $x \in 2^{\omega}$, $\delta > 0$

$$\mathrm{K}(w) = \min\{|y| \,|\, U(y) = w\}$$

$$\mathrm{K}_\delta(x) = \inf\{\mathrm{K}(q) \,|\, q \in \mathbb{Q}, |x - q| < \delta\}$$

$$\dim(x) = \liminf_{\delta \to 0^+} \frac{\mathrm{K}_\delta(x)}{\log(1/\delta)}.$$

$$\dim(A) = \sup_{x \in A} \dim(x)$$

(and similarly for Dim, $\mathrm{Dim}(x) = \limsup_{\delta \to 0^+} \frac{\mathrm{K}_\delta(x)}{\log(1/\delta)}$)

Ways to generalize

Why do we effectivize?

- To quantify
- Partial randomness
- Geometric measure theory

Ways to generalize

Why do we effectivize?

- To quantify
- Partial randomness
- Geometric measure theory

Ways to generalize effective dimension

- Make it more precise, avoid infinite dimension cases
- Use different resource-bounds, avoid dimension 0 spaces
- Relativize to compare those effectivizations

The gauge function ingredient

To avoid infinite dimension

- A **gauge function** is a continuous, nondecreasing function from $[0,\infty)$ to $[0,\infty)$ that vanishes only at 0.
- A **gauge family** is a one-parameter family $\varphi = \{\varphi_s \,|\, s \in (0,\infty)\}$ of gauge functions φ_s satisfying for $s > t$, $\varphi_s(\delta) = o(\varphi_t(\delta))$ as $\delta \to 0^+$

Definition

$$H^{s,\varphi}(E) = \lim_{\delta\to 0} \inf_{\mathcal{U} \text{ is a } \delta\text{-cover of } E} \sum_{U\in\mathcal{U}} \varphi_s(\mathrm{diam}(U))$$

$$\dim^{\varphi}(E) = \inf \{ s \,|\, H^{s,\varphi}(E) = 0 \} .$$

They generalize $\theta_s(\delta) = \delta^s$ in Hausdorff dimension.

The gauge function ingredient

To avoid infinite dimension

- A **gauge function** is a continuous, nondecreasing function from $[0, \infty)$ to $[0, \infty)$ that vanishes only at 0.
- A **gauge family** is a one-parameter family $\varphi = \{\varphi_s \,|\, s \in (0, \infty)\}$ of gauge functions φ_s satisfying for $s > t$, $\varphi_s(\delta) = o(\varphi_t(\delta))$ as $\delta \to 0^+$

Definition

$$H^{s,\varphi}(E) = \lim_{\delta \to 0} \inf_{\mathcal{U} \text{ is a } \delta\text{-cover of } E} \sum_{U \in \mathcal{U}} \varphi_s(\mathrm{diam}(U))$$

$$\dim^{\varphi}(E) = \inf \left\{ s \,|\, H^{s,\varphi}(E) = 0 \right\}.$$

They generalize $\theta_s(\delta) = \delta^s$ in Hausdorff dimension.
We can define φ-gales $d : 2^{<\omega} \to [0, \infty)$ with

$$d(w)\varphi_s(2^{-|w|}) = (d(w0) + d(w1))\varphi_s(2^{-|w|-1})$$

The resource-bound ingredient

- **Finite-State dimension**: base dependent, randomness is dimension 1 (normality), gambling and compression, no universality
- **p-dimension**: only gambling, complexity classes (NP), close to qp-dimension, no universality
- **pspace-dimension**: gambling and compression, no universality
- **dim**: gambling and compression, universality

They each have distinctive properties

The relativization ingredient

Except for the finite state case, all definitions relativize to any oracle $B \subseteq \mathbb{N}$.

Point-to-set principles

Theorem (Lutz Lutz 2018)

Let $A \subseteq 2^\omega$. *Then*

$$\dim_{\mathrm{H}}(A) = \min_{B \subseteq \mathbb{N}} \dim^B(A).$$

Theorem (Lutz Lutz 2018)

Let $A \subseteq 2^\omega$. *Then*

$$\dim_{\mathrm{P}}(A) = \min_{B \subseteq \mathbb{N}} \mathrm{Dim}^B(A).$$

Resource-bounded point-to-set principles

qp = quasi-polynomial time, $2^{(\log n)^k}$

Theorem (Lutz Lutz M 2021)

Let $A \subseteq 2^\omega$. Then

$$\dim_{\mathrm{qp}}(A) = \min_{g \in \mathrm{qp}} \dim_{\mathrm{p}}^{g}(A).$$

Theorem (Lutz Lutz M 2021)

Let $A \subseteq 2^\omega$ and $\Gamma < \Delta$. Then

$$\dim_{\Delta}(A) = \min_{g \in \Delta} \dim_{\Gamma}^{g}(A).$$

Application of point to set principles to fractal geometry: projection formula

Theorem (Marstrand 1954)

Let $E \subseteq \mathbb{R}^2$ be an analytic set with $\dim_{\mathrm{H}}(E) = s$. Then for almost every $\theta \in (0, 2\Pi)$, $\dim_{\mathrm{H}}(p_\theta E) = \min\{s, 1\}$

Application of point to set principles to fractal geometry: projection formula

Theorem (Marstrand 1954)

Let $E \subseteq \mathbb{R}^2$ be an analytic set with $\dim_{\mathrm{H}}(E) = s$. Then for almost every $\theta \in (0, 2\Pi)$, $\dim_{\mathrm{H}}(p_\theta E) = \min\{s, 1\}$

It does not hold for arbitrary E (assuming CH). Recently an extension using PSP

Theorem (N.Lutz Stull 2018)

Let $E \subseteq \mathbb{R}^2$ be an arbitrary set with $\dim_{\mathrm{H}}(E) = \dim_{\mathrm{P}}(E) = s$. Then for almost every $\theta \in (0, 2\Pi)$, $\dim_{\mathrm{H}}(p_\theta E) = \min\{s, 1\}$

Application of point to set principles to fractal geometry: projection formula

Theorem (Marstrand 1954)

Let $E \subseteq \mathbb{R}^2$ be an analytic set with $\dim_{\mathrm{H}}(E) = s$. Then for almost every $\theta \in (0, 2\Pi)$, $\dim_{\mathrm{H}}(p_\theta E) = \min\{s, 1\}$

It does not hold for arbitrary E (assuming CH). Recently an extension using PSP

Theorem (N.Lutz Stull 2018)

Let $E \subseteq \mathbb{R}^2$ be an arbitrary set with $\dim_{\mathrm{H}}(E) = \dim_{\mathrm{P}}(E) = s$. Then for almost every $\theta \in (0, 2\Pi)$, $\dim_{\mathrm{H}}(p_\theta E) = \min\{s, 1\}$

Further extension in (Stull 2021)

Other

- (N.Lutz 2021) Intersection formula (extension from Borel to all)
- (N.Lutz Stull 2020) results on Furstenberg sets
- (Slaman 2021) The Hausdorff dimensions of co-analytic sets are not carried by their closed subsets
- (Lutz 2021) There are Hamel bases ($\mathbb{R}$ over $\mathbb{Q}$) with any positive Hausdorff dimension

Looking at other separable spaces

- Where can we effectivize dimension?
    - We can define Kolmogorov complexity/ effectivize Hausdorff measure if we have a **separator** (countable dense set)

Looking at other separable spaces

- Where can we effectivize dimension?
  - We can define Kolmogorov complexity/ effectivize Hausdorff measure if we have a **separator** (countable dense set)

Definition (Kolmogorov complexity of x at precision δ)

Let (X, ρ) be a separable metric space and let $D \subseteq X$ be a countable dense set (fix $f : 2^{<\omega} \twoheadrightarrow D$)

$$\mathrm{K}_\delta(x) = \inf\left\{\mathrm{K}(w) \,\middle|\, w \in 2^{<\omega}, \rho(x, f(w)) < \delta\right\}$$

Looking at other separable spaces

Definition

The *algorithmic dimension* and strong algorithmic dimension of a point $x \in X$ is

$$\dim(x) = \liminf_{\delta \to 0^+} \frac{\mathrm{K}_\delta(x)}{\log(1/\delta)},$$

$$\mathrm{Dim}(x) = \limsup_{\delta \to 0^+} \frac{\mathrm{K}_\delta(x)}{\log(1/\delta)}.$$

Looking at other spaces: gauged dimension

Definition

The *φ-gauged algorithmic dimension* and strong algorithmic dimension of a point $x \in X$ is

$$\dim^{\varphi}(x) = \inf \left\{ s \,\middle|\, \liminf_{\delta \to 0^+} 2^{\mathrm{K}_{\delta}(x)} \varphi_s(\delta) = 0 \right\},$$

and the *φ-gauged* of x is

$$\mathrm{Dim}^{\varphi}(x) = \inf \left\{ s \,\middle|\, \limsup_{\delta \to 0^+} 2^{\mathrm{K}_{\delta}(x)} \varphi_s(\delta) = 0 \right\},$$

Looking at other spaces: gauged dimension

Definition

The *φ-gauged algorithmic dimension* and *strong algorithmic dimension* of a point $x \in X$ is

$$\dim^{\varphi}(x) = \inf \left\{ s \,\middle|\, \liminf_{\delta \to 0^+} 2^{\mathrm{K}_\delta(x)} \varphi_s(\delta) = 0 \right\},$$

and the *φ-gauged* of x is

$$\mathrm{Dim}^{\varphi}(x) = \inf \left\{ s \,\middle|\, \limsup_{\delta \to 0^+} 2^{\mathrm{K}_\delta(x)} \varphi_s(\delta) = 0 \right\},$$

$$d(w)\varphi_s(2^{-|w|}) = (d(w0) + d(w1))\varphi_s(2^{-|w|-1})$$

General Point-to-set principles

Let (X, ρ) be a separable metric space, φ a gauge family

Theorem (Lutz Lutz M 2022)

Let $A \subseteq X$. Then

$$\dim_{\mathrm{H}}^{\varphi}(A) = \min_{B \subseteq \mathbb{N}} \sup_{x \in A} \dim^{\varphi, B}(x).$$

Theorem (Lutz Lutz M 2022)

Let $A \subseteq X$. Then

$$\dim_{\mathrm{P}}^{\varphi}(A) = \min_{B \subseteq \mathbb{N}} \sup_{x \in A} \mathrm{Dim}^{\varphi, B}(x).$$

An exercise: the Hilbert cube

- Let (X, ρ) be a compact separable metric space, let (a_n) be an ℓ_2 sequence of positive real numbers
- Let $H(X; (a_n))$ be the set of infinite sequences of X together with the metric

$$d_a(x, y) = \left(\sum_n a_n^2 \rho(x_n, y_n)^2 \right)^{1/2}$$

- $H(X; (a_n))$ has infinite Hausdorff dimension
- What is the right gauged dimension for it?

An exercise: the Hilbert cube

- Try to get for each $x \in H(X; (a_n))$, i.o. δ,

$$\varphi_s(\delta) < 2^{-\mathrm{K}_\delta(x)}$$

An exercise: the Hilbert cube

- Try to get for each $x \in H(X; (a_n))$, i.o. δ,

$$\varphi_s(\delta) < 2^{-\mathrm{K}_\delta(x)}$$

- E.g. $H([0, 1]; (1/n))$

$$\mathrm{K}_{2^{-k}}(x) \leq 2^{kc}$$

An exercise: the Hilbert cube

- Try to get for each $x \in H(X;(a_n))$, i.o. δ,

$$\varphi_s(\delta) < 2^{-\mathrm{K}_\delta(x)}$$

- E.g. $H([0,1];(1/n))$

$$\mathrm{K}_{2^{-k}}(x) \leq 2^{kc}$$

- $\varphi_s(\delta) = 2^{-1/\delta^s}$ (the power-exponential scale)

The hyperspace

- Let (X, ρ) be a separable metric space
- Let $\mathcal{K}(X)$ be the set of nonempty compact subsets of X together with the Hausdorff metric $\mathrm{dist}_{\mathrm{H}}$ defined as follows

$$\mathrm{dist}_{\mathrm{H}}(U, V) = \max\left\{\sup_{x\in U} \rho(x, V), \sup_{y\in V} \rho(y, U)\right\}.$$

$(\rho(a, B) = \inf\{\rho(a, b) \,|\, b \in B\})$

Relationship of the dimensions of E and $\mathcal{K}(E)$

McClure (1995 and 1996) has several results relating Hausdorff and packing dimensions of a set E and $\mathcal{K}(E)$ for

- E self-similar
- E σ-compact

Relationship of the dimensions of E and $\mathcal{K}(E)$

McClure (1995 and 1996) has several results relating Hausdorff and packing dimensions of a set E and $\mathcal{K}(E)$ for

- E self-similar
- E σ-compact

$\mathcal{K}(E)$ has infinite dimension, a different gauge family is needed

A result by McClure

Theorem (McClure 1995)

*Let $E \subseteq X$ be σ-**compact**. Let $\psi_s(\delta) = 2^{-1/\delta^s}$. Then*

$$\dim_{\mathrm{P}}^{\psi}(\mathcal{K}(E)) \geq \dim_{\mathrm{P}}(E).$$

A result by McClure

Theorem (McClure 1995)

*Let $E \subseteq X$ be σ-**compact**. Let $\psi_s(\delta) = 2^{-1/\delta^s}$. Then*

$$\dim_{\mathrm{P}}^{\psi}(\mathcal{K}(E)) \geq \dim_{\mathrm{P}}(E).$$

We aim to extend the theorem to other E and to other gauge families beside the canonical one.

A result by McClure

Theorem (McClure 1995)

*Let $E \subseteq X$ be σ-**compact**. Let $\psi_s(\delta) = 2^{-1/\delta^s}$. Then*

$$\dim_{\mathrm{P}}^{\psi}(\mathcal{K}(E)) \geq \dim_{\mathrm{P}}(E).$$

We aim to extend the theorem to other E and to other gauge families beside the canonical one.

Definition

The *jump* of a gauge family φ is the family $\widetilde{\varphi}$ given $\widetilde{\varphi}_s(\delta) = 2^{-1/\varphi_s(\delta)}$.

For the canonical gauge family $\theta_s(\delta) = \delta^s$, $\widetilde{\theta}_s(\delta) = 2^{-1/\delta^s}$

Hyperspace packing dimension theorem

Theorem (LLM 2022)

Let $E \subseteq X$ be an analytic set, and let φ be a gauge family, then

$$\dim_{\mathrm{P}}^{\widetilde{\varphi}}(\mathcal{K}(E)) \geq \dim_{\mathrm{P}}^{\varphi}(E).$$

Proof ideas: where we use PSP

- By the general point-to-set principle, let A be an oracle such that
$$\dim_{\mathrm{P}}^{\widetilde{\varphi}}(\mathcal{K}(E)) = \sup_{L \in \mathcal{K}(E)} \mathrm{Dim}^{\widetilde{\varphi},A}(L),$$

Proof ideas: where we use PSP

- By the general point-to-set principle, let A be an oracle such that
$$\dim_{\mathrm{P}}^{\widetilde{\varphi}}(\mathcal{K}(E)) = \sup_{L \in \mathcal{K}(E)} \mathrm{Dim}^{\widetilde{\varphi},A}(L),$$
- We recursively construct a single compact set $L \in \mathcal{K}(E)$ (i.e., a single point in the hyperspace $\mathcal{K}(E)$) so that it has high Kolmogorov complexity at infinitely many precisions, relative to oracle A.
$$\mathrm{Dim}^{\widetilde{\varphi},A}(L) > s$$

Proof ideas: where we use PSP

- By the general point-to-set principle, let A be an oracle such that
$$\dim_{\mathrm{P}}^{\widetilde{\varphi}}(\mathcal{K}(E)) = \sup_{L \in \mathcal{K}(E)} \mathrm{Dim}^{\widetilde{\varphi},A}(L),$$
- We recursively construct a single compact set $L \in \mathcal{K}(E)$ (i.e., a single point in the hyperspace $\mathcal{K}(E)$) so that it has high Kolmogorov complexity at infinitely many precisions, relative to oracle A.
$$\mathrm{Dim}^{\widetilde{\varphi},A}(L) > s$$

$$\mathrm{K}_{\delta}(L) > -\log \widetilde{\varphi_s}(\delta)$$

Proof ideas: where we use PSP

- By the general point-to-set principle, let A be an oracle such that
$$\dim_{\mathrm{P}}^{\widetilde{\varphi}}(\mathcal{K}(E)) = \sup_{L\in\mathcal{K}(E)} \mathrm{Dim}^{\widetilde{\varphi},A}(L),$$
- We recursively construct a single compact set $L \in \mathcal{K}(E)$ (i.e., a single point in the hyperspace $\mathcal{K}(E)$) so that it has high Kolmogorov complexity at infinitely many precisions, relative to oracle A.
$$\mathrm{Dim}^{\widetilde{\varphi},A}(L) > s$$
$$\mathrm{K}_{\delta}(L) > -\log \widetilde{\varphi_s}(\delta)$$
- For E compact, we can reach $\mathrm{Dim}^{\widetilde{\varphi},A}(L) \geq s$ for $s = \dim_{\mathrm{P}}^{\varphi}(E)$

Open questions

- Can we get $\mathrm{Dim}^{\widetilde{\varphi},A}(L) \geq \dim_{\mathrm{P}}^{\varphi}(E)$ for E more general than compact?
- Is there a more general hyperspace Hausdorff dimension theorem? $\dim_{\mathrm{H}}^{\widetilde{\varphi}}(\mathcal{K}(E))$ vs $\dim_{\mathrm{H}}^{\varphi}(E)$ for interesting E
- Are (the complexity or the good properties of) the two oracles in the PTSP related to hyperspace dimension theorems?

References

- Jack H. Lutz and Neil Lutz, Who asked us? How the theory of computing answers questions about analysis, Ding-Zhu Du and Jie Wang (eds.), Complexity and Approximation: In Memory of Ker-I Ko, pp. 48-56, Springer, 2020.
- Jack H. Lutz and Elvira Mayordomo, Algorithmic fractal dimensions in geometric measure theory. In Vasco Brattka and Peter Hertling (eds.), Handbook of Computability and Complexity in Analysis, Springer-Verlag (2021).
- Jack H. Lutz, Neil Lutz, and Elvira Mayordomo, Extending the Reach of the Point-to-Set Principle. STACS 2022 `https://arxiv.org/abs/2004.07798`