Subclasses of effective supermartingales: completeness phenomenon

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Does a given subclass of left-c.e. supermartingales define 1-randomness.

Motivation-summary

1. Can computable objects define 1-randomness;
2. Can 1-randomness be decomposed;
3. KL-random vs 1-random;

Motivation—can computable object define 1-randomness

- What is randomness?
- Randomness $\Leftrightarrow$ "No pattern".
- Strings with some "pattern": 010101010101, 01100011100000111111.

Motivation—can computable object define 1-randomness

- Effective randomness $\Leftrightarrow$ No effective pattern.
- Effective pattern: a sequence $(V_n \subseteq 2^{<\omega} : n \in \omega)$ of uniformly c.e. sets (with $[V_n] \supseteq [V_{n+1}]$) such that $m(V_n) \leq 2^{-n}$ (known as *Martin-Löf test*).

Motivation—can computable object define 1-randomness

Definition 1

A real $X \in 2^\omega$ is *Martin-Löf random* (also called *1-random*) if no Martin-Löf test $(V_n : n \in \omega)$ succeed on X. i.e., $X \notin \bigcap_n [V_n]$.

- Many definitions of effective randomness turn out to be equivalent (to 1-randomness).
- For example, X is 1-random iff there is no left-c.e. supermartingale M succeeding on X (i.e., $\limsup_n M(X \upharpoonright n) < \infty$).
- Here a *left-c.e. supermartingale* is a non decreasing computable array $(M[t] : t \in \omega)$ of supermartingales such that $\lim_{t \to \infty} M[t](\sigma) = M(\sigma)$ exists for all $\sigma \in 2^{<\omega}$.

Motivation—can computable object define 1-randomness

- Unfortunately all definitions of 1-randomness concern c.e.ness, which is dissatisfactory since it is supposed to be an effective randomness notion. Numerous definitions that try not to use c.e.ness are given such as:
  1. Schnorr randomness: the reals on which no Schnorr test succeed (a Schnorr test is a Martin-Löf test with $m(V_n)$ being computable);
  2. Kurtz randomness: the reals that cannot be contained in any measure 0 effectively closed subset of 2^ω;
  3. computable randomness: the reals on which no computable martingale succeed.
- But none of them are as strong as 1-randomness (1-randomness implies them but not vice versa).

Motivation—can computable object define 1-randomness

Is there a complexity notion weaker than left-c.e.ness yet makes the supermartingales (of that complexity) define 1-randomness.

Question 2

Or is there a class of left-c.e. supermartingales whose behaviour is somewhat "predictable" defining 1-randomness.

Motivation—can 1-randomness be decomposed

- If a left-c.e. supermartingale $M : 2^{<\omega} \to \mathbb{R}^{\geq 0}$ succeeds on X, is it because of its betting strategy of outcome or its strategy of money allocation;
- Here we say M is *i-sided* at $\sigma \in 2^{<\omega}$ iff $M(\sigma i) \geq M(\sigma^\frown(1-i))$.

Question 3 (Kasterman?)

Can we decompose M into M_0, M_1 (meaning $M_0 + M_1$ succeeds on all reals on which M succeeds) such that M_i is i-sided.

Motivation—KL-randomness vs 1-randomness

Will explain later.

Question 4

Can "natural" subclass of left-c.e. supermartingale define 1-randomness?

kastergale

- For a computable martingale M, we could know (computably) whether $M(\sigma 1) \geq M(\sigma 0)$.
- For a function $p :\subseteq 2^{<\omega} \to 2$, we say M is *p-sided* if for every $\sigma \in dom(p)$, M is $p(\sigma)$-sided at σ, and for every $\sigma \notin dom(p)$, M is both 0-sided, 1-sided at σ.

kastergale

Definition 5 (kastergale)

For left-c.e. supermartingale M, we say M is *partially-computably-sided* (known as *kastergale*) iff:

for some partial computable function p, $M[t]$ is $p[t]$-sided.

i.e., For each $\sigma \in 2^{<\omega}$, M has only one chance to decide its sidedness at σ and before it makes that decision, it has to be both $0, 1$-sided at σ.

muchgale

Definition 6 (muchgale)

A supermartingale M is *(l, i)-betting* if for every σ such that $|\sigma| \equiv i \mod (l)$, we have $M(\sigma) \geq \max\{M(\sigma 0), M(\sigma 1)\}$. i.e., M does not bet at certain steps. A *muchgale* is a left-c.e. supermartingale that is (l, i)-betting for some l, i.

Questions and known results

- Kasterman wondered if kastergales define 1-randomness (i.e., whether for every non-1-random real X there is a kastergale succeeding on X) [Downey, 2012];
- Hitchock asked the same question with respect to a subclass of kastergale where the biased proportion $M(\sigma i)/M(\sigma)$ is Σ^0_1 function;
- Barmpalias, Fang and Lewis-Pye [Barmpalias et al., 2020] considered single-sided (p-sided with $p \equiv i$ for some $i \in 2$) left-c.e. supermartingales whose bias is non decreasing and showed that they do not define 1-randomness.
- Muchnick [Muchnik, 2009] considered $(2, i)$-betting left-c.e. supermartingales and showed that they do not define 1-randomness.

Conclusion

Theorem 7 ([Barmpalias and Liu, 2021])

The union of kastergales and muchgales does not define 1-randomness. i.e., there is a non-1-random real X on which no kastergale or muchgale succeed.

Conclusion

Our analysis shows that

$$\text{If a reasonable subclass of left-c.e. supermartingales defines 1-randomness, it almost means a single member of that class can do so.} \tag{2.1}$$

Formalize (2.1)

- A class of *supermartingale-approximations* is a set $\mathcal{M}$ of supermartingale sequences $M[\leqslant t] = (M[0], \cdots, M[t])$.
- $\mathcal{M}$ is *non decreasing* iff: $M[t]$ dominates $M[t-1]$;
- $\mathcal{M}$ is *scale-closed* iff: iff for every $M[\leqslant t] \in \mathcal{M}$, every $c > 0$, $cM[\leqslant t] \in \mathcal{M}$.
- We say $\mathcal{M}$ is *subsequence-closed* iff for every $M[\leqslant t] \in \mathcal{M}$, every $t_0 < \cdots < t_{s-1} \leq t$, $(M[t_0], \cdots, M[t_{s-1}]) \in \mathcal{M}$.

Formalize (2.1)

- We say $\mathcal{M}$ is *homogeneous* iff, roughly speaking, looking at $\mathcal{M}$ on a cone $[\rho]^{\preceq}$ is the same as that on $[\emptyset]^{\preceq}$.
- Homogeneous, subsequence-closed, scale-closed, Π^0_1 class:
  kastergales;
  given l, $\{(l,i)\text{-betting supermartingales} : i < l\}$;
  muchgale.
- In (2.1), by reasonable, we mean scale-closed, subsequence-closed, homogeneous and Π^0_1.

- An $\mathcal{M}$-*gale* is: a ω-sequence $M[<\omega]$ such that $M[\leqslant t] \in \mathcal{M}$ for all $t \in \omega$ and $\lim_{t\to\infty} M[t](\sigma)$ exists for all $\sigma \in 2^{<\omega}$.

A game

Whether computable $\mathcal{M}$-gales define 1-randomness $\hookrightarrow$

Whether Alice (controlling the Martin-Löf test) wins against Baby (controlling members of $\mathcal{M}$) in the following game.

A game

The finite version of this game:

Definition 8 ((c, n, k)-$\mathcal{M}$-game)

At each round $t \in \omega$:
Alice: enumerates $\sigma \in 2^n$;
Baby: presents $M_j[t]$ (for each $j < k$) such that:

- $\sum_j M_j[t](\hat{\sigma}) \geq 1$ for some $\hat{\sigma} \preceq \sigma$ (for all $\sigma \in A[t]$);
- $M_j[\leqslant t] \in \mathcal{M}$ for all $j < k$.

Alice wins if: $\sum_j M_j[t](\emptyset) \geq c$.

Let A denote the set of σ Alice enumerates when she wins.

A game

- Roughly speaking, if Alice has a winning strategy for (c, n, k)-$\mathcal{M}$-game with an arbitrary small cost $m(A)$, then $\mathcal{M}$ does not define 1-randomness.
- Let$\mathcal{M} = \cup_l \mathcal{M}_l$ where $\mathcal{M}_l \subseteq \mathcal{M}_{l+1}$ is Π^0_1 (uniformly in l), non decreasing, scale-closed, subsequence-closed and homogeneous.

Claim 9

If for every $l, k \in \omega, \varepsilon > 0, c < 1$, Alice has a winning strategy for (c, n, k)-$\mathcal{M}_l$-game (for some n) such that $m(A) \leq \varepsilon$, then computable $\mathcal{M}$-gales do not define 1-randomness.

The constant game

Let $a, \Delta, \delta > 0, n, k \in \omega$:

Definition 10 (constant $(a, \Delta, \delta, n, k)$-$\mathcal{M}$-game)

At each round $t \in \omega$:
Alice: $\sigma \in 2^n$,
Baby: $M_j[t]$ such that:

- $\sum_j M_j[t](\sigma) \geq 1$ (for all $\sigma \in A[t]$);
- $M_j[\leqslant t] \in \mathcal{M}$ for all $j < k$.
- $\sum_j M_j[t](\rho) \leq 1 + \delta$ for all $\rho \in 2^{\leq n}$.

Alice wins if:

- (type-(a)) $1 - \sum_j M_j[t](\emptyset) \leq (1 - m(A[t]))/a$; or
- (type-(b)) for some $\sigma_0, \sigma_1 \in A[t]$, $||\vec{M}[t](\sigma_0) - \vec{M}[t](\sigma_1)||_1 \geq \Delta$

constant $\mathcal{M}$-game vs $\mathcal{M}$-game

- "$\sum_j M_j[t](\sigma) \geq 1$" vs "$\sum_j M_j[t](\hat{\sigma}) \geq 1$ for some $\hat{\sigma} \preceq \sigma$";
- $\sum_j M_j[t](\rho) \leq 1 + \delta$;
- dynamic winning criterion "$1 - \sum_j M_j[t](\emptyset) \leq (1 - m(A[t]))/a$" vs "$\sum_j M_j[t](\emptyset) \geq c$"
- for some $\sigma_0, \sigma_1 \in A[t]$, $||\vec{M}[t](\sigma_0) - \vec{M}[t](\sigma_1)||_1 \geq \Delta$

Reduce to constant game

- Roughly speaking, if Alice could win the constant $\mathcal{M}$-game (for $k = 1$) with $m(A) < 1$, then she could win the $\mathcal{M}$-game (for all k) with an arbitrary small $m(A)$.
- Let$\mathcal{M}$ be non decreasing and homogeneous.

Claim 11

If for every $a > 0$, there exist $\Delta, \delta > 0, n \in \omega$ such that Alice has a winning strategy for the constant $(a, \Delta, \delta, n, 1)$-$\mathcal{M}$-game with $m(A) < 1$, then for every $\varepsilon > 0, c < 1, k \in \omega$ there is an n such that Alice has a winning strategy for (c, n, k)-$\mathcal{M}$-game such that $m(A) \leq \varepsilon$.

Reduce to constant game

Proof.

See [Barmpalias and Liu, 2021].
section 2.1-2.2 (dynamic winning criterion),
section 2.3 (restricting Baby's action),
section 4.2 (type-(b) winning criterion),
section 4.3 (reduce to $k = 1$). □

Reduce to constant game

- For kastergale or (l, i)-betting supermartingale-approximation, it's easy to win the constant game (for $k = 1$), thus Theorem 7 follows.
- Winning the constant game (for $k = 1$) is the only part of the proof where we take advantage of sidedness and (l, i)-betting.

Completeness phenomenon

- Moreover, if $\mathcal{M}$ could define 1-randomness, then (for some $a > 0$, for every $\Delta, \delta > 0$, every $n \in \omega$) Alice does not have a winning strategy for the constant $(a, \Delta, \delta, n, 1)$-$\mathcal{M}$-game so that $m(A) < 1$.
- This almost means that a single member of $\mathcal{M}$ (the one Baby used against Alice) could define 1-randomness.
- With that said, this is not a concrete proof of (2.1), but a strong evidence.

A close look at iteration argument

Let $c_i \leq 1, \varepsilon_i \geq 0, n_i \in \omega$ for each $i < 2$.

Claim 12

If (for each $i < 2$) Alice has a winning strategy for (c_i, n_i, k)-$\mathcal{M}$-game such that $m(A) \leq \varepsilon_i$. Then Alice has a winning strategy for $(c_0c_1, n_0 + n_1, k)$-$\mathcal{M}$-game such that $m(A) \leq \varepsilon_0\varepsilon_1$.

Proof.

- In $(c_0c_1, n_0 + n_1, k)$-$\mathcal{M}$-game, invoke winning strategy of (c_0, n_0, k)-$\mathcal{M}$-game.
- But when the strategy tells you to enumerate $\rho \in 2^{n_0}$, instead of enumerating it, play the winning strategy of (c_1, n_1, k)-$\mathcal{M}$-game at the board $[\rho]^{\preceq} \cap 2^{n_0+n_1}$.
- Hopefully, the sub-game will forces $\sum_{j<k} M_j(\rho)[t] \geq c_1$.

□

More efficient winning strategy

For $\mathcal{M} = \{(2, i)\text{-betting supermartingale-approximation}\}$, Alice can win the (c, n, k)-$\mathcal{M}$-game with a cost $m(A) \approx 1/2$ (for sufficiently large n); moreover, this is optimal:

Lemma 13 ([Barmpalias and Liu, 2022])

- *There is a real X with $dim_H(X) = 1/2$ such that there is no $(2, i)$-betting left-c.e. supermartingale succeeding on X.*
- *For every real X with $dim_H(X) < 1/2$, every $i \in 2$, there is a $(2, i)$-betting left-c.e. supermartingale succeeding on X;*

More efficient winning strategy

Alice can win the (c, n, k)-$\mathcal{M}$-game (with $c = 1, n = 2$) with a cost $m(A) \leq \frac{3}{4}$. Thus, let $dim_P(X)$ denote the *packing dimension* of X, namely $\limsup_n K(X \restriction n)/n$.

Theorem 14 ([Barmpalias and Liu, 2022])

There is a real $X \in 2^\omega$ on which no $(2, i)$-betting left-c.e. supermartingale succeeds for all $i < 2$ such that $dim_P(X) \leq 1 - \frac{1}{2}\log_2(4/3)$.

Given a subclass $\mathcal{M}$ of left-c.e. supermartingales and $d \geq 0$,

Question 15

Is there a real X with $dim_H(X) \leq d$ (resp. $dim_P(X) \leq d$) such that there is no member of $\mathcal{M}$ succeeding on X.

Question 16

Is there a winning strategy of Alice on the (c, n, k)-$\mathcal{M}$-game (when n is sufficiently large) such that $m(A) \leq \exp(-O(1)n)$?

Many thanks

Is there any question?

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