Cousin’s lemma in second-order arithmetic

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Cousin's lemma

Cousin's lemma is a principle of compactness:

- Let $\mathcal{I} = \{I_x : x \in [0, 1]\}$ be a collection of intervals, open in $[0, 1]$, with $x \in I_x$ for all x. Then $\mathcal{I}$ has a finite sub-cover.

Compare with *countable compactness*:

- If $\mathcal{I} = \{I_n : n \in \mathbb{N}\}$ is a countable collection of intervals, open in $[0, 1]$, and $[0, 1] = \bigcup_n I_n$, then $\mathcal{I}$ has a finite sub-cover.

Recall that countable compactness is equivalent to WKL_0 [Simpson IV.1.2].

A reformulation

Definition

- A *gauge* is a function $\delta \colon [a, b] \to (0, \infty)$.
- A *δ-fine partition* is a tagged partition $a = x_0 \leqslant \xi_1 \leqslant x_1 \leqslant \xi_2 \leqslant x_2 \leqslant \cdots \leqslant \xi_n \leqslant x_n = b$ such that for all $i = 1, \ldots, n$,
$$\delta(\xi_i) > x_i - x_{i-1}.$$

This generalises the *mesh size* of a partition: if δ is a constant, then a δ-fine partition is one with mesh size $< \delta$.

Lemma (Cousin's lemma)

For every gauge δ there is a δ-fine partition.


aaaaaaaaannnd...


aaaaaaaaaannnd...
why should
I care?
SEDUFF

The gauge integral

Also known as the Henstock-Kurzweil integral; equivalent to the Denjoy integral and the Perron integral.

Definition

Let $f\colon [a,b] \to \mathbb{R}$ be a function.

$$\int_a^b f = r$$

if for every $\epsilon > 0$ there is a gauge δ such that for any δ-fine tagged partition P, the associated partial sum is within ϵ of r.

This clearly generalises the Riemann integral.

An example

Let $1_{\mathbb{Q}}$ be Dirichlet's function.

Fix an enumeration $\langle q_k : k \in \mathbb{N} \rangle$ of $\mathbb{Q} \cap [0,1]$.

Given $\epsilon > 0$, let

- $\delta(q_k) = \epsilon 2^{-k}$;
- for irrational $x \in [0,1]$, $\delta(x) = 1$.

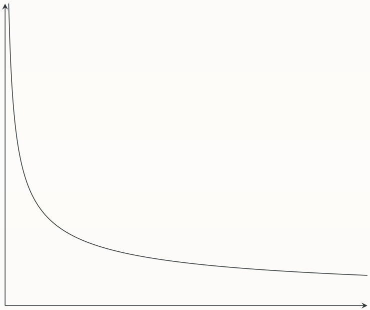
This shows that

$$\int_0^1 1_{\mathbb{Q}} = 0$$

(but it is not Riemann integrable).

Another example

Let $f(x) = 1/\sqrt{x}$.



The reason f is not Riemann integrable is that no matter how small a mesh size δ, we can choose a tag ξ with $f(\xi) \gg \delta$. This is prevented if $\delta(\xi) \ll f(x)$.

Nice properties

The gauge integral:

- Extends the Lebesgue integrals;
- If f is differentiable on $[a, b]$ then f' is gauge integrable and

$$\int_a^b f' = f(b) - f(a).$$

- No improper integrals: if for all $\epsilon > 0$, f is gauge integral on $[a + \epsilon, b]$, then f is gauge integrable on $[a, b]$ and $\int_a^b f = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b f$.

Cousin's lemma is required for the notion not to be vacuous.

The strength of Cousin’s lemma

> *The similarities between the proof of the existence of δ-fine tagged partitions of $[a, b]$ and the proof (at least one of the standard proofs) that the interval $[a, b]$ is a compact set are evident. This is no accident — the two statements are actually equivalent.*
>
> —*Russell A. Gordon, The use of tagged partitions in elementary real analysis*

Gordon's evidence

Cousin's lemma implies:

- The intermediate value theorem;
- A continuous function on a closed interval is bounded;
- A continuous function on a closed interval obtains a maximum;
- A continuous function on a closed interval is unifomrly continuous;
- A continuous function on a closed interval is Riemann integrable;
- Mean value inequalities.

Note that most are equivalent to WKL_0.

For example

Theorem

If f is differentiable on $[a, b]$ and $f' > 0$ on $[a, b]$ then f is increasing on $[a, b]$.

Proof.

By applying the argument on each sub-interval, it suffices to show that $f(b) > f(a)$.

For every $\xi \in [a, b]$, since $f'(\xi) > 0$, there is some $\delta(\xi) > 0$ such that if $\xi \in [x, y]$ and $|y - x| < \delta(\xi)$ then $f(y) - f(x) > 0$.

If (x_k, ξ_k) is a δ-fine partition then $f(x_0) < f(x_1) < \cdots < f(x_n)$. $\square$

Second-order?

Cousin's lemma is a statement of *third*-order arithmetic. In this context, Normann and Sanders showed that it is equivalent to full second-order arithmetic.

Within second-order arithmetic, we need to restrict ourselves to classes of countably-coded functions. We will look at classes of Borel functions.

The proof of Cousin's lemma

Let δ be a gauge on $[0, 1]$.

- An interval I is *good* if it can be a part of a δ-fine partition: there is some $z \in I$ with $\delta(z) > |I|$.
- Let T be the tree of all bad binary rational intervals $I = [k2^{-n}, (k+1)2^{-n}]$.
- If T is finite then the children of the leaves of T form a δ-fine partition of $[0, 1]$.
- T cannot be infinite: say the intersection of an infinite path is a singleton $\{z^*\}$; since $\delta(z^*) > 0$, any sufficiently small interval on the path is good.

Conclusion:

- Cousin's lemma for Borel functions is provable in $\Pi^1_1 - \mathsf{CA}_0$.

Continuous functions

Theorem

Cousin's lemma for continuous functions is equivalent to WKL_0.

Proof from WKL_0**:**

Since δ is continuous, if I is good then there is a rational witness.
Hence T is Π^0_1. $\square$

Easier proof from WKL_0**:**

In WKL_0, a continuous gauge δ has a minimum, which is positive.
So any partition with sufficiently small mesh size suffices. $\square$

Continuous functions: the reversal

Suppose that M is a model of RCA_0 in which WKL_0 fails.

- There is a closed set $C \subseteq [0,1]$ with code in M which is nonempty "in the real world", but M thinks that C is empty.
- Let $\delta(x) = \frac{1}{2}d(x, C)$; then δ is not a gauge, but M thinks that it is.
- In the real world, there is no δ-fine partition, as such a partition cannot cover any points in C.
- But being a δ-fine partition is absolute, so there are none in M.

Baire class 1

Recall: A *Baire class 1* function is a pointwise limit of continuous functions.

Theorem
Cousin's lemma for Baire class 1 functions is equivalent to ACA_0.

Proof from ACA_0:

- A Baire class 1 function pulls back open sets to $\mathbf{\Sigma}^0_2$ sets.
- $\varnothing''$ can tell whether a given Σ^0_2 set is empty or not.
- Hence in ACA_0, the tree T of the proof of Cousin's lemma exists.

Nonstandard model complication: the set of leaves of T is M-finite. We need to choose a point $\xi_I \in I$ with $\delta(\xi_I) > |I|$ for each child I of a leaf. In ACA_0, we can choose points from a sequence of nonempty Σ^0_2 sets. $\square$

Baire class 1: the reversal

Let M be a model in which ACA_0 fails.

- There is a left-c.e. real z^* which is not in M.
- Let $\delta(x) = \frac{1}{2}|z^* - x|$. Then M thinks that δ is a gauge.
- M does not have a code of δ as a continuous function: but it has a definition of δ as a Baire class 1 function.
- There cannot be a δ-fine partition, as no such partition can cover z^*. So again, there are none in M.

Nonstandard model complication: we use Σ^0_1 induction to argue that if P is a partition then z^* must lie in one of the P-intervals. We need z^* to be left-c.e. rather than any Δ^0_2 real.

Borel functions

Theorem

The following are equivalent (modulo some induction):

1. *Cousin's lemma for Baire class 2 functions*
2. *Cousin's lemma for Borel functions*
3. ATR_0.

Proof from ATR_0+induction:

- In $\mathsf{ATR}_0+\Sigma^1_1$-induction: if $T \subseteq 2^{<\omega}$ is Π^1_1 and infinite, then it has a path.

Note that T need not exist in the model.

- In ATR_0+a bit more than Σ^1_1-induction: if $T \subseteq 2^{<\omega}$ is Π^1_1 and has bounded height, then it exists.
- Choosing a tag for each leaf: use Σ^1_1-choice (or Σ^1_1-induction).

$\square$

Cantor space

For simplicity, work in Cantor space. Cousin's lemma for Baire class 2 functions implies:

- If $f\colon 2^\omega \to \mathbb{N}$ is Baire class 2, then there is a finite set $P \subset 2^\omega$ such that
$$2^\omega = \bigcup_{x \in P} [x \upharpoonright f(x)].$$

(Call this an *f-fine cover*.)

Note that a Baire class 2 function $f\colon 2^\omega \to \mathbb{N}$ is one for which $f(x)$ is uniformly computable from x''.

Baire class 2 functions and iterated jumps

- Suppose that M is a standard model of ACA_0 in which Cousin's lemma for Baire class 2 functions holds. Then for all computable α, $\varnothing^{(\alpha)} \in M$.

Proof for $\alpha = \omega$:

Suppose that $\varnothing^{(\omega)} \notin M$. Recall that $\varnothing^{(\omega)} = \bigoplus_n \varnothing^{(n)}$.

- For each $x \neq \varnothing^{(\omega)}$, there is some least $n = n_x$ such that $x^{[n]} \neq \varnothing^{(n)}$.
- For such x, there is some least $k = k_x$ such that $x^{[n]}(k) \neq \varnothing^{(n)}(k)$.
- The relation $y = \varnothing^{(n)}$ is Π^0_2. Hence x'' can compute n_x and k_x.
- So x'' computes some m sufficiently large so that $x(m) \neq \varnothing^{(\omega)}(m)$. Let $f(x) = m + 1$; so $\varnothing^{(\omega)} \notin [x \upharpoonright f(x)]$.
- There is no f-fine cover, since $\varnothing^{(\omega)}$ really exists.
- As usual, this is absolute for M.

$\square$

Baire class 2 functions: the reversal

Let *M* be a model of ACA_0 in which ATR_0 fails. Then there is some *M*-ordinal δ^* such that in *M*, there is no jump hierarchy along δ^*.

Note that for all $\beta < \delta^*$, by ACA_0, there is at most one jump hierarcy along β in *M*. Denote it by $\varnothing^{(\beta)}$.

Let

$$\mathscr{I} = \left\{\beta < \delta^* \,:\, \varnothing^{(\beta)} \text{ exists in } M\right\}.$$

This is an initial segment of δ^* (of course, not in *M*). ACA_0 implies that $\mathscr{I}$ does not have a greatest element.

Baire class 2 functions: the reversal

Let $x \in M$.

- There is a least $\beta = \beta_x \in \mathscr{I}$ such that $x^{[\beta]} \neq \varnothing^{(\beta)}$.
- There is a least $k = k_x$ such that $x^{[\beta]}(k) \neq \varnothing^{(\beta)}(k)$.
- x'' can find these.
- So x'' computes some $m = f(x)$ such that $[x \restriction m]$ does not contain $\varnothing^{(\gamma)}$ for any $\gamma > \beta$ in $\mathscr{I}$.
- If P is an f-fine cover, let $\beta^* = \max\{\beta_x : x \in P\} + 1$.
- Then $\beta^* \in \mathscr{I}$ but P does not cover $\varnothing^{(\beta^*)}$.

Thank you