The coding power of product of partitions

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- If a P_1 instance I_1 encode a P_0 instance I_0 (meaning every solution of I_1 computes a solution of I_0), what can we say about I_0, I_1?

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- Instance of RT^n_k: k-coloring of n-tuples of integers $C : [\omega]^n \to k$;

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- $\mathsf{RT}^1_{k+1} \not\leq_{soc} \mathsf{RT}^1_k$ (Patey [7]);
- When $n > 1$, does $\mathsf{RT}^n_{k+1} \leq_{soc} \mathsf{RT}^n_k$ (Patey and Monin)?

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General picture: if there is no obvious way that P_1 encode P_0, then it can't.

Connection to RM and CCR

- (RM): P_1 implies P_0 (over RCA) ———— P_0 can be solved by invoking P_1;
- $\mathsf{RT}^n_2 \leftrightarrow \mathsf{RT}^n_k$———— invoking RT^n_2 k times solves RT^n_k.

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- $\mathsf{RT}^n_2 \leftrightarrow \mathsf{RT}^n_k$——— invoking RT^n_2 k times solves RT^n_k.
- (CCR): coding randomness in P.
- Every RT^1_k instance admit a solution that doe not compute any 1-random real (Kjos-Hanssen [5]).

Product of colorings

- $(\mathsf{RT}^1_2)^r$-Instance: $(C_0, \cdots, C_{r-1})$ where $C_s \in 2^\omega$;
- $(\mathsf{RT}^1_2)^r$-solution: $(G_0, \cdots, G_{r-1})$ where $G_s \subseteq \omega \wedge |G_s| = \infty$ is monochromatic for C_s for all $s < r$.

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Question 1

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Question 1

Does $\mathsf{RT}^1_3 \leq_{soc} (\mathsf{RT}^1_2)^r$?

Theorem 2 (L. [6])

- $\mathsf{RT}^1_3 \not\leq_{soc} (\mathsf{RT}^1_2)^{<\omega}$; *i.e.,*
- *There is a 3-coloring $C : \omega \to 3$ such that for every $r \in \omega$, every finitely many 2-colorings $C_0, \cdots, C_{r-1}$, there is a solution to $(C_0, \cdots, C_{r-1})$ that does not compute any solution to C.*
- *Moreover, C can be $\emptyset^{(\omega)}$-computable.*

Ludovic Patey independently obtained an answer of Question 1.

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Question 3

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i.e., it requires the 3-coloring and the 2-colorings in Theorem 2 to be Δ^0_2.
Actually, we have

Theorem 4 (L.[6])

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- A similar lemma shows that if a 2-coloring $\hat{C}$ uniformly encode a 2-coloring $\tilde{C}$, then it must be the case that $\hat{C}$ comptably "copies" $\tilde{C}$.
- How complex does the class Q has to be so as to satisfy the cross constraint.
- How weak can the witness be when the Π^0_1 class doesn't satisfy the cross constraint. Weakening the witness in certain ways will address Theorem 4. Such strengthening of the lemma is a type of basis theorem for Π^0_1 class with certain combinatorial constraint. We introduce several variants of this type of basis theorem among them many are open.

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The frame work

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In the end, take the common element of $p_0, p_1, \cdots$ (which exists by compactness), so it satisfies all requirements.

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- In this particular theorem, it suffices to prove:

Lemma 5

For every tuple of Turing functionals $\{\Psi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in 2^r}$, every tuple of colors $\{j_{\boldsymbol{k}}\}_{\boldsymbol{k}\in 2^r}$, there is a $\boldsymbol{k}^* \in 2^r$, a solution $(G_0, \cdots, G_{r-1})$ in color $\boldsymbol{k}^*$ of $(C_0, \cdots, C_{r-1})$ such that $\Psi_{\boldsymbol{k}^*}^{(G_0,\cdots,G_{r-1})}$ is not a solution of C in color $j_{\boldsymbol{k}^*}$.

Proof of Lemma 5

- Observe the behavior of $\{\Psi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in 2^r}$ by wondering which $\tilde{C} \in 3^\omega$ is encoded via $\{\Psi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in 2^r}$ by some $(\hat{C}_0, \cdots, \hat{C}_{r-1}) \in (2^\omega)^r$ (as in Lemma 5). i.e.,

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- Consider the set Q of such $(\tilde{C}, \hat{C})$ that $\tilde{C} \in 3^\omega$ is encoded by $\hat{C} = (\hat{C}_0, \cdots, \hat{C}_{r-1}) \in (2^\omega)^r$ via $\{\Psi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in 2^r}$ (as in Lemma 5).

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- Note that Q is a Π^0_1 class.

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- $\tilde{C}^0, \tilde{C}^1$ are almost disjoint while $\hat{C}^0_s, \hat{C}^1_s$ are not for all $s < r$.
- Pick up a vector of infinite sets $(G_0, \cdots, G_{r-1})$ so that it is in color $\boldsymbol{k}^*$ of $\hat{C}^0, \hat{C}^1, (C_0, \cdots, C_{r-1})$ for some $\boldsymbol{k}^*$.

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- Because $\Psi^{(G_0,\cdots,G_{r-1})}_{\boldsymbol{k}^*}$ is a solution in color $j_{\boldsymbol{k}^*}$ of both $\tilde{C}^0, \tilde{C}^1$. Thus it must be finite since $\tilde{C}^0, \tilde{C}^1$ are almost disjoint.

What we need in Lemma 5

For two k-colorings C_0, C_1, we say C_0, C_1 are *almost disjoint* if for every $j < k$, $C_0^{-1}(j) \cap C_1^{-1}(j)$ is finite.

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Let $Q \subseteq 3^\omega \times (2^\omega)^r$ be a Π^0_1 class that has full projection on 3^ω.

Lemma 6 (L.[6])

There exists $(X^0, Y^0), (X^1, Y^1) \in Q$ such that: X^0, X^1 are almost disjoint and Y^0_s, Y^1_s are not almost disjoint for all $s < r$. Moreover, $(X^0, Y^0) \oplus (X^1, Y^1)$ is $\emptyset'$-computable.

Proof.

Combinatorial forcing and paring argument. □

Question 7

Given a 2-coloring $\tilde{C}$. Suppose for some 2-coloring $\hat{C}$, some Turing functionals $\{\Psi_s\}_{s<r}$, every solution G of $\hat{C}$ computes a solution of $\tilde{C}$ via some Ψ_s, what can we say about $\tilde{C}$ and $\hat{C}$.

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- We say $\hat{C}$ *computably copies* $\tilde{C}$ if There are computable functions $f:\omega\rightarrow\omega$ and $g:\omega\rightarrow 2$ with $f^{-1}(\hat{n})$ being finite for all $\hat{n}$ such that $(\forall n)\hat{C}(n)=g(n)+\tilde{C}(f(n))mod(2)$.

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- We say $\hat{C}$ *computably homogeneously copies* $\tilde{C}$ if in addition, g is constant.
- We verify the intuition for most of $\tilde{C}$. Let $\tilde{C}$ be hyperimmune and admits no Δ^0_2 solution.

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- *The 2-coloring $\hat{C}$ computably copies $\tilde{C}$.*

Let $\tilde{C}$ be hyperimmune relative to $\mathcal{O}^m$ for all m.

Theorem 10

Suppose every solution of $\hat{C}$ computes a solution of $\tilde{C}$, then for some m, some $\mathcal{O}^m$-computable infinite set Z, $\hat{C} \upharpoonright Z$ $\mathcal{O}^m$-computably copies $\tilde{C}$.

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- Just like Lemma 5 boils down to prove the cross constraint can not be satisfied by certain closed set $Q \subseteq 3^\omega \times (2^\omega)^r$,
- The above theorems boils down to the following lemma, saying that if certain close set $Q \subseteq 2^\omega \times 2^\omega$ satisfies a stronger version of the cross constraint, then it is satisfied in a regular way.
- For a collection $P \subseteq 2^\omega$, we say P is almost disjoint if $\bigcap_{X \in P} X^{-1}(i)$ is finite for all $i \in 2$.

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i.e., there is a $\hat{Q}$ with $\hat{Q} \cap O \subseteq Q \cap O$ and with $proj_0((Q \cap O) \setminus \hat{Q})$ being meger such that $\hat{Q}$ is defined as following. For some functions $f : \omega^* \to \omega, g : dom(f) \to 2$ with $f^{-1}(\hat{n})$ being finite for all $\hat{n}$, we have: for every $(X, Y) \in O$, $(X, Y) \in \hat{Q}$ if and only if $Y(n) = g(n) + X(f(n))$ for all $n \in dom(f)$.

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There exists an clopen set O such that for some functions $f : \omega^* \to \omega$, $g : dom(f) \to 2$, we have $\Gamma(X)(n) = g(n) + X(f(n))\ mod(2)$ for all $n \in dom(f)$ and all $X \in O$.

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- Clearly corollary 12 is the infinite version of the following observation: if $\Gamma : 2^n \to 2$ preserves disjoint (meaning $\Gamma(V) = \{0, 1\}$ whenever V is disjoint), then there is an $m < n$, an $i \in 2$ such that $\Gamma(\sigma) = \sigma(m) + i\ mod(2)$ for all $\sigma \in 2^n$.

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- For more results in this spirit, see e.g. regularity theorems on automorphism of the boolean algebra $\mathcal{P}(\omega)/fin$ (Velikovic[9],Shelah[8] Chapter IV).

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Question 13

- If $\hat{C}$ uniformly encode $\tilde{C}$ (meaning for some tuple of Turing functionals $\{\Psi_s\}_{s<r}$, for every solution G of $\hat{C}$, Ψ_s^G is a solution of $\tilde{C}$ for some $s < r$). What can we say about $\tilde{C}, \hat{C}$.

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Above Π^0_1 class.

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(†)for every $(X^0, Y^0), (X^1, Y^1) \in Q,$ if X^0, X^1 are almost disjoint, then Y^0_s, Y^1_s are almost disjoint for some $s < r$.

Above Π^0_1 class—— a game theoretic view.

- When $r = 1$, Q does not exist and the reason is "finite":

  there are three mutually disjoint 3-coloring,

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- Bob wins if for every $m \neq m'$, $\tilde{C}_m, \tilde{C}_{m'}$ being almost disjoint implies $\hat{C}_m, \hat{C}_{m'}$ being almost disjoint.

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- Bob wins if for every $m \neq m'$, $\tilde{C}_m, \tilde{C}_{m'}$ being almost disjoint implies $\hat{C}_m, \hat{C}_{m'}$ being almost disjoint.
- Bob has a winning strategy.
- Does Bob has a winning strategy without looking at the game history?

Proposition 14

If Q is Σ^1_1, then Q does not satisfy $(\dagger)$.

Proof.

Combine Cohen forcing and the proof of Lemma 6 □

An ultrafilter construction of Q.

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If there exists a non principal ultrafilter on ω*, then there exists a function* $\Gamma : 3^\omega \to (2^\omega)^2$ *such that for every two almost disjoint* $X^0, X^1 \in 3^\omega$*,* $\Gamma(X^0)_s, \Gamma(X^1)_s$ *are almost disjoint for some* $s < 2$*.*

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Proof.

Let $\Gamma(X) = (\emptyset, \emptyset), (\emptyset, \omega), (\omega, \emptyset)$ respectively depending on for which $j \in 3$, $X^{-1}(j) \in \mathcal{U}$.

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Let $\Gamma(X) = (\emptyset, \emptyset), (\emptyset, \omega), (\omega, \emptyset)$ respectively depending on for which $j \in 3$, $X^{-1}(j) \in \mathcal{U}$.
Where $\emptyset$ represents the 2-coloring Z such that $Z^{-1}(1) = \emptyset$ and similarly for ω. □

A Π^1_1 definition.

Moreover, Jonathan showed that the assertion "there exists a Π^1_1 set $Q \subseteq 3^\omega \times (2^\omega)^3$ with full projection on 3^ω satisfying (†)" is consistent with ZFC.

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If $\mathbf{V} = \mathbf{L}$, then there exists a Π^1_1 set $Q \subseteq 3^\omega \times (2^\omega)^3$ with full projection on 3^ω satisfying (†).

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Proposition 16 (Jonathan)

If $\mathbf{V} = \mathbf{L}$*, then there exists a* Π^1_1 *set* $Q \subseteq 3^\omega \times (2^\omega)^3$ *with full projection on* 3^ω *satisfying (†).*

Proof.

In $\mathbf{L}$, we can construct a Σ^1_2 non principal ultrafilter $\mathcal{U}$ on ω. Suppose $\mathcal{U}$ is defined by $\exists Z_0 \forall Z_1 \varphi(Z_0, Z_1, Z)$. Combine with the construction of Proposition 15 and leave one component of $(2^\omega)^3$ for Z_0 in the definition of $\mathcal{U}$.

□

The set theoretic proof strength

- Let ECC(r) denote the assertion " there is a set $Q \subseteq 3^\omega \times (2^\omega)^r$ satisfying the constraint ($\dagger$)";

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- Let ECC(r) denote the assertion " there is a set $Q \subseteq 3^\omega \times (2^\omega)^r$ satisfying the constraint (†)";
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The set theoretic proof strength

- Let ECC(r) denote the assertion " there is a set $Q \subseteq 3^\omega \times (2^\omega)^r$ satisfying the constraint (†)";
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- By Proposition 15, over ZF, $\mathrm{EU}(\omega) \to \mathrm{ECC}(r) \to \mathrm{ECC}(r+1)$ for all $r > 3$.

Question 17

Are implications (set theoretic) in $\mathrm{EU}(\omega) \to \mathrm{ECC}(r) \to \mathrm{ECC}(r+1)$ strict?

Reducing Theorem 4 to an improvement of Lemma 6

As Theorem 2 is reduced to Lemma 6, Theorem 4 boils down to the following improvement of Lemma 6:

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Let $Q \subseteq 3^\omega \times (2^\omega)^r$ be a Π^0_1 class that has full projection on 3^ω; let C be a 3-coloring that is some sort of hyperimmune.

Lemma 18 (L.[6])

There exist $(X^0, Y^0), (X^1, Y^1) \in Q$ such that:

- *X^0, X^1 are almost disjoint and Y^0_s, Y^1_s are not almost disjoint for all $s < r$;*
- *moreover, C is still hyperimmune relative to $(X^0, Y^0) \oplus (X^1, Y^1)$.*

Basis theorem for Π^0_1 class with constraint (†)

Clearly Lemma 18 is a type of basis theorem with the additional constraint. Since the corresponding hyperimmune basis theorem for Π^0_1 class says:

Proposition 19

For every non empty Π^0_1 class $Q \subseteq 2^\omega$, every hyperimmune function $f : \omega \to \omega$, there is a $X \in Q$ such that f is hyperimmune relative to X.

Basis theorem for Π_1^0 class with constraint (†)

In general,

Question 20

Suppose $\mathcal{W}$ is a collection of which the basis theorem for Π_1^0 class holds, does the (†)-constraint version basis theorem holds?

Basis theorem for Π^0_1 class with constraint (†)

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Question 20

Suppose $\mathcal{W}$ is a collection of which the basis theorem for Π^0_1 class holds, does the (†)-constraint version basis theorem holds?

We have the (†)-constraint version of Cone avoidance and low basis theorem.

Lemma 21

There exist $(X^0, Y^0), (X^1, Y^1) \in Q$ such that:

- X^0, X^1 *are almost disjoint and* Y^0_s, Y^1_s *are not almost disjoint for all* $s < r$*;*
- $(X^0, Y^0) \oplus (X^1, Y^1)$ *is low and does not compute a given Turing degree.*

Basis theorem for Π_1^0 class with general constraint

Note that if we look at general constraint, then it is possible that the constraint version of some basis theorem is no longer true.

Basis theorem for Π^0_1 class with general constraint

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Taking cone avoidance as an example:

Proposition 22

There exists a non empty Π^0_1 class $Q \subseteq 2^\omega$ such that for every $X^0, X^1 \in Q$, if $X^0 \neq^* X^1$, then $X^0 \oplus X^1 \geq_T \emptyset'$.

Proof.

Note that there is a non empty Π^0_1 class $Q \subseteq 2^\omega$ such that for every $X \in Q$, if X, as a set, is infinite, then $X \geq_T \emptyset'$.

□

Yet another basis theorem

Question 23

Given two incomputable Turing degree $D_0 \not\geq_T D_1$, a non empty Π^0_1 class $Q \subseteq 2^\omega$, does there exist a $X \in Q$ such that $X \not\geq_T D_0$ and $D_0 \oplus X \not\geq_T D_1$?

Strong cone avoidance of non hyperarithmetic degree

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Proposition 24

The problem $(\mathrm{RT}^1_2)^\omega$ admit strong cone avoidance for non hyperarithmetic Turing degree.

Encoding RT_3^1

Question 25

Is there a 3-coloring C not encoded by any product of infinitely many 2-colorings?

That is: is there a 3-coloring C such that for any sequence of 2-colorings $C_0, C_1, \cdots$, there exists a solution $(G_0, G_1, \cdots)$ to $(C_0, C_1, \cdots)$ such that $(G_0, G_1, \cdots)$ does not compute any solution to C.

Thank you for attending. Is there any question(s)?

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