

Reduction games, provability, and compactness

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Joint work with Damir D. Dzhamalov and Denis R. Hirschfeldt

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Background on Π_2^1 -problems

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We denote such problems by **P** and **Q**.

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Theorem (Folklore/Wang)

If

$$\text{ACA}_0 \vdash \forall X [\Theta(X) \rightarrow \exists Y \Delta(X, Y)]$$

where Θ and Δ are arithmetic, then there is an $n \in \omega$ such that

$$\text{ACA}_0 \vdash \forall X [\Theta(X) \rightarrow \exists Y \in \Sigma_n^{0,X} \Delta(X, Y)].$$

Models of second-order arithmetic

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An ω -model satisfies ACA_0 iff it is a jump ideal, that is, iff it is closed under the Turing jump.

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We say that P is **Weihrauch reducible** to Q , $P \leq_W Q$, if there are Turing functionals Φ and Ψ such that, for every instance X of P , the set Φ^X is an instance of Q , and for every solution \hat{Y} to $\hat{X} = \Phi^X$, the set $Y = \Psi^{X \oplus \hat{Y}}$ is a solution to X .

Examples of Π_2^1 -problems

We write $[X]^n$ for the collection of n -element subsets of X . A **k -coloring** of $[X]^n$ is a map $c : [X]^n \rightarrow k$. A coloring of $[X]^2$ is **stable** if $\lim_{y \in X} c(x, y)$ exists for all $x \in X$. $H \subseteq X$ is **homogeneous** for c if there exists an i such that $c(s) = i$ for all $s \in [H]^n$. $L \subseteq X$ is **limit-homogeneous** for $c : [X]^2 \rightarrow k$ if there exists an i such that $\lim_{y \in L} c(x, y) = i$ for all $x \in L$.

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- ▶ D_k^2 : every stable k -coloring of $[\mathbb{N}]^2$ has an infinite limit-homogeneous set.

Reductions between example problems

Theorem (Jockusch)

Let $n \geq 2$. Then

Every computable instance of $RT_{<\infty}^n$ has a Π_n^0 solution.

There is a computable instance of RT_2^n with no Σ_2^0 solution.

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Theorem (Cholak, Jockusch, and Slaman)

$RCA_0 + RT_2^2 \not\vdash RT_{<\infty}^2$.

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We also have that $\text{RCA}_0 \vdash \text{RT}_k^1$ for each k . Therefore $\text{RT}_{<\infty}^1$ is true in every ω -model of RCA_0 . However,

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Equivalently, $\text{RT} \leq_\omega \text{RT}_2^3$ but $\text{RCA}_0 + \text{RT}_2^3 \not\vdash \text{RT}$.

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- ▶ If the game never ends, then Player 1 wins.
- ▶ If a player is unable to make a move, their opponent wins.

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- ▶ Player 2 then either plays an $(X_0 \oplus \dots \oplus X_{n-1})$ -computable solution to X_0 and wins, or plays an $(X_0 \oplus \dots \oplus X_{n-1})$ -computable instance Y_n of Q

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We have that $RT_{<\infty}^n \leq_{gW} RT_2^n$ and $RT \leq_{gW} RT_2^3$.

Winning $G(Q \rightarrow P)$

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*If $P \leq_{\omega} Q$ then Player 2 has a winning strategy for $G(Q \rightarrow P)$.
Otherwise, Player 1 has a winning strategy for $G(Q \rightarrow P)$.*

Definition

We say that P is **generalized Weihrauch reducible** to Q and write $P \leq_{gW} Q$, if Player 2 has a computable winning strategy for $G(Q \rightarrow P)$.

We have that $RT_{<\infty}^n \leq_{gW} RT_2^n$ and $RT \leq_{gW} RT_2^3$.

If Player 2 has a winning strategy for $G(Q \rightarrow P)$ in at most $n + 1$ many moves, then we write $P \leq_{\omega}^n Q$, and likewise for gW .

Generalized Weihrauch reductions between example problems

Theorem (Hirschfeldt and Jockusch)

Let $n \geq 3$, $j \geq 1$, and m be such that $n + (j - 1)(n - 2) < m \leq n + j(n - 2)$. Then $\text{RT}_k^m \leq_{\text{gW}}^{j+1} \text{RT}_k^n$, but $\text{RT}_k^m \not\leq_{\omega}^j \text{RT}_k^n$. Therefore $\text{RT} \not\leq_{\omega}^j \text{RT}_2^3$ for all j , although $\text{RT} \leq_{\omega} \text{RT}_2^3$.

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Patey showed that for $n \geq 3$, the $RT_k^n \leq_{\omega}^2 RT_j^n$ for $j < k$, but $RT_k^n \not\leq_{\omega}^1 RT_j^n$. For $n = 2$, the least m such that $RT_k^n \leq_{\omega}^m RT_j^n$ approaches ∞ as k increases.

Extending Π_2^1 -problems

In our definition of Π_2^1 -problems, we required that instances and solutions be subsets of ω . We can extend this notion more generally as follows: Let \mathcal{M} be an \mathcal{L}_1 -structure with domain $|M|$. For $S \subseteq |M|$, we write (\mathcal{M}, S) for the \mathcal{L}_2 -structure with first-order part \mathcal{M} and second-order part S . For an \mathcal{L}_1 -structure \mathcal{M} , an **\mathcal{M} -instance** of P is an $X \subseteq |M|$ such that $(\mathcal{M}, \{X\}) \models \Theta(X)$, and a **solution** to X is a $Y \subseteq |M|$ such that $(\mathcal{M}, \{X, Y\}) \models \Psi(X, Y)$.

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Let \mathcal{N} be a \mathcal{L}_1 -structure. For $X_0, \dots, X_n \in |N|$, let $\mathcal{N}[X_0, \dots, X_n] = (\mathcal{N}, S)$, where S consists of all subsets of $|N|$ that are Δ_1^0 -definable from parameters in $|N| \cup \{X_0, \dots, X_n\}$.

The game $G^\Gamma(Q \rightarrow P)$

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The Γ -reduction game $G^\Gamma(Q \rightarrow P)$ is defined as follows:

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Let Γ be a set of \mathcal{L}_2 -formulas and P and Q be Π_2^1 -problems. The **modified Γ -reduction game $\hat{G}^\Gamma(Q \rightarrow P)$** is defined as follows:

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Results from modified games

Proposition

Let Γ be a consistent extension of Δ_1^0 -comprehension by Π_1^1 -formulas. Let P and Q be Π_2^1 -problems. If $\Gamma \vdash Q \rightarrow P$, then Player 2 has a winning strategy for $G^\Gamma(Q \rightarrow P)$. Otherwise, Player 1 has a winning strategy for $\hat{G}^{\Gamma+Q}(Q \rightarrow P)$.

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For our main result, we need the mild extra assumption that Γ proves the existence of a universal Σ_1^0 -formula.

The main result

Theorem

Let Γ satisfy the conditions. Let P and Q be Π_2^1 -problems. If $\Gamma \vdash Q \rightarrow P$, then there is an n such that Player 2 has a winning strategy for $\hat{G}^\Gamma(Q \rightarrow P)$ that ensures victory in at most n many moves. Otherwise, Player 1 has a winning strategy for $\hat{G}^{\Gamma+Q}(Q \rightarrow P)$.

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Corollary

Let $\Gamma = \text{RCA}_0 +$ all Π_1^1 -formulas true over ω . If $P \not\leq_\omega^n Q$ for all n , then $\Gamma \not\vdash Q \rightarrow P$.

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Example

Let $Q = \text{RT}_2^2$ and $P = \text{RT}_{<\infty}^2$. Patey showed that $\text{RT}_{<\infty}^2 \not\leq_\omega^n \text{RT}_k^2$ for all n, k . Therefore $\Gamma \not\vdash \text{RT}_2^2 \rightarrow \text{RT}_{<\infty}^2$.

Essential lemma

For $n \in \omega$, let $\Theta_n(\mathbf{e}_0, \dots, \mathbf{e}_n, X_0, \dots, X_n, Y_0, \dots, Y_n)$ be a formula asserting that

if X_0 is a P-instance then $(Y_0 = \Phi_{\mathbf{e}_0}^{X_0} \wedge (\text{either } Y_0 \text{ is a solution to } X_0$
or $(Y_0 \text{ is a Q-instance and if } X_1 \text{ is a solution to } Y_0 \text{ then } (Y_1 = \Phi_{\mathbf{e}_1}^{X_0 \oplus X_1}$
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 $\dots (Y_n = \Phi_{\mathbf{e}_n}^{X_0 \oplus \dots \oplus X_n} \wedge Y_n \text{ is a solution to } X_0)) \dots)$.

If $\Gamma \vdash Q \rightarrow P$, then there exists an $n \in \omega$ such that

$$\Gamma \vdash \forall X_0 \exists \mathbf{e}_0, Y_0 \forall X_1 \exists \mathbf{e}_1, Y_1 \dots \forall X_n \exists \mathbf{e}_n, Y_n \\ \Theta_n(\mathbf{e}_0, \dots, \mathbf{e}_n, X_0, \dots, X_n, Y_0, \dots, Y_n).$$

Extending generalized Weihrauch reducibility

Definition

We say that P is **generalized Weihrauch reducible to Q over Γ** and write $P \leq_{\text{gW}}^{\Gamma} Q$, if Player 2 has a computable (i.e., Δ_1^0), winning strategy for $G^{\Gamma}(Q \rightarrow P)$.

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An example: limit-homogeneous sets

Recall that a coloring of $[X]^2$ is **stable** if $\lim_{y \in X} c(x, y)$ exists for all $x \in X$. $L \subseteq X$ is **limit-homogeneous** for $c : [X]^2 \rightarrow k$ if there exists an i such that $\lim_{y \in L} c(x, y) = i$ for all $x \in L$.

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We have the principles:

SRT_k²: every stable k -coloring of $[\mathbb{N}]^2$ has an infinite homogeneous set.

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Theorem (Chong, Lempp, and Yang)

SRT₂² and D₂² are equivalent over RCA₀.

An example: limit-homogeneous sets cont.

Theorem (Dzhafarov)

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Note that an instance of LH includes the color i to which the set is limit-homogeneous.

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Theorem

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Recall that $\text{SRT}_2^2 \leq_{\text{gW}} D_2^2$.

The following questions remain open:

Is $\text{SRT}_2^2 \leq_{\text{gW}}^{\text{RCA}_0} D_2^2$? Is $\text{LH} \leq_{\text{gW}}^{\text{RCA}_0} D_2^2$?