

# Sacks Splitting Theorem Revisited

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# Motivation

- ▶ One of the fundamental results of computability theory is Sacks' Splitting Theorem:

## Theorem (Sacks, 1963)

*If  $A$  is c.e. and  $\emptyset <_T C \leq_T \emptyset'$  then there exists a c.e. splitting  $A_1 \sqcup A_2 = A$  with  $C \not\leq_T A_i$  for  $i \in \{1, 2\}$ .*

- ▶ This fundamental result
  1. Showed that there were no minimal c.e. degrees,
  2. Ushered in one form of the infinite injury method (although it is not an infinite injury argument, but finite injury of "unbounded type".)
  3. Was the basis of huge technical progress on the c.e. degrees.

## For Example

### Theorem (Robinson, 1971)

*Everything c.e. If  $C <_T A$  and  $C$  low, then  $A = A_1 \sqcup A_2$  with  $C \oplus A_1 \upharpoonright_T C \oplus A_2$ . Hence every c.e. degree split over every lesser low c.e. degree.*

Robinson's Theorem was very influential in that it showed how to use "lowness+c.e." a theme we follow to this day.

### Theorem (Lachlan, 1975)

*There exist  $c < a$  such that  $a$  does not split over  $c$ .*

Affected the architecture of computability theory thereafter. E.g. definability, decidability etc. Invented the  $\mathbf{0}'''$  method to prove this result. Harrington improved Lachlan's Theorem to have  $\mathbf{a} = \mathbf{0}'$ .

## Re-examining this

- ▶ Lots of questions can be asked about the 60 year old result.
- ▶ For example: “How unbounded is the finite injury?”
- ▶ In recent work, not talked about here this can be quantified:

### Theorem (Ambos-Spies, Downey, Monath, Ng)

*If  $A$  is c.e. then  $A$  can always be split into a pair of totally  $\omega^2$ -c.a. c.e. sets. (Here totally  $\omega^2$  c.a. means that if  $f \leq_T A_i$  is total, then it is  $\omega^2$ -c.a. in the Downey-Greenberg classification.)*

- ▶ Sacks' proof only gives  $\omega^\omega$ -c.a..
- ▶ Earlier Selwyn and I showed that this is tight:

### Theorem (Downey and Ng, 2018)

*There is a c.e. degree  $\mathbf{a}$  such that if  $\mathbf{a}_1 \vee \mathbf{a}_2 = \mathbf{a}$  then  $\mathbf{a}_i$  is not totally  $\omega$ -c.a. for  $i \in \{1, 2\}$ .*

- ▶ Lots of similar questions remain:
- ▶ For example:

## Question

1. *Is the Ambos-Spies et. al. theorem valid if we also add cone avoidance?*
2. *What about adding lowness?*
3. *What can be said about the degrees which are joins of totally- $\omega$ -c.a. c.e. degrees? (This is a definable class.)*

# Re-re-examining Sacks

- ▶ Here the question Guohua and I looked at:

## Question

*Is the natural analog for avoiding **lower** cones valid?*

- ▶ The answer is **no**.

## Theorem (Downey, Wu)

*There are c.e. sets  $B <_T A$  such that whenever  $A_1 \sqcup A_2 = A$  is a c.e. splitting, then for some  $i \in \{1, 2\}$ ,  $A_i \leq_T B$ .*

- ▶ We remark that the degree analog is true because either  $\mathbf{a}$  splits over  $\mathbf{b}$ , or  $\mathbf{b}$  cups  $\mathbf{a}_2$  to  $\mathbf{a}$  and we can then choose  $\mathbf{b} < \mathbf{a}_1 < \mathbf{a}$  by Sacks's Density Theorem.  $\mathbf{b}$  for some  $\mathbf{a}_2$  (i.e. lower cone avoidance happens)

# The Proof

- ▶ The proof is non-trivial, and uses the  $\mathbf{0}'''$  method.
- ▶ We need  $B \leq_T A$ , say  $\Xi^A = B$ .
- ▶ Requirements  $B \leq_T A$  and

$$R_e : W_e \sqcup V_e = A \rightarrow (\exists \Gamma_e (\Gamma_e^B = W_e) \vee \exists \Delta_e (\Delta_e^B = V_e)).$$

$$N_e : \Phi^B \neq A.$$

- ▶ We will define a rather complicated priority tree  $PT$  and there meet  $R_e$  at nodes  $\tau$ , with outcomes  $\infty <_L f$ .
- ▶ The procedures  $\Gamma_e, \Delta_e$  built via axioms as usual.
- ▶ We meet  $N_e$  at nodes  $\sigma$ .

# The Basic Module

- ▶ Drop the “e”
- ▶ One  $N = N_j$  at a  $\sigma$  below  $\tau^\wedge_\infty$  for  $R = R_e$ .
- ▶ The overall goal of  $N$  is to have  $\ell(j, s) = \max\{z \mid \Phi_j^B \upharpoonright z = A \upharpoonright z[s]\} > y$ , for **some**  $y$  and put  $y$  into  $A$  whilst preserving  $B \upharpoonright \varphi_j(y)$ .
- ▶ The obvious problem is that if we put  $y$  into  $A_{s+1}$  then assuming  $\tau^\wedge_\infty \prec TP$ ,  $y$  will enter one of  $W$  or  $V$ .
- ▶ Now, depending on which we believe we are proving,  $(\Gamma^B = W) \vee (\Delta^B = V)$ , this would then entail putting something into  $B$ , i.e. something below  $\gamma(y, s)$  or  $\delta(y, s)$ .
- ▶ On the other hand, if we are monitoring only  $\Delta$ , say, and  $y$  enters  $W$  and not  $V$ , we would not care.



- ▶  $\sigma$  We will either prove that  $W$  is computable (finite in the basic module) (the  $\Sigma_3^0$  outcome) or if no  $\sigma$  does this, then  $\tau$  will prove  $\Delta^B = A$ . (the  $\Pi_3^0$  outcome)
- ▶ In the general construction we build  $\Gamma_\sigma^B = W$ .
- ▶ That is, we are “favouring”  $V$  at  $\sigma$ , in cooperation with  $\tau$ .
- ▶  $N$  picks a follower  $x$  with a trace  $t_0 = \delta(x, s)$ .
- ▶ The strategy runs in cycles. At each stage we will have a trace  $t_n = \delta(x, s)$ .
- ▶ The **goal** is to try to have
  1. Either  $\delta(x, s) > \varphi_j(x, s)$  when  $\ell(j, s) > x$ , or
  2. Put something into  $A$  which meets  $N_j$  and went into  $W$ .
- ▶ In the first case if  $x$  entered  $V$ , we could still correct  $\Delta^B$  using  $\delta(x, s)$  without injuring  $\Phi_j^B(x) \neq A(x)[s+1]$  as  $\delta(x, s) > \varphi(x, s)$ .
- ▶ Now it **might** be that neither occurs. Then
  1. Everything we use (i.e. the  $t_n$ 's) to attack  $N$  will enter  $V$  and not  $W$ . (Thus  $W$  is computable (in fact empty).)
  2.  $\varphi(x, s) \rightarrow \infty$  and hence  $\Phi_j^B(x) \uparrow$ . Note that  $\Delta$  will be partial, but that's okay, as  $\sigma$  gives a proof that  $W$  is computable (or  $\Gamma_\sigma^B = W$ , more generally).

- ▶ We hit  $\sigma$  and see  $\ell(j, s) > t_n (> x)$ .
- ▶ **Case 1.**  $t_n = \delta(x, s) > \varphi_j(x, s)$ .  
**Action** Put  $x$  into  $A_{s+1} - A_s$ . This will meet  $N$ . At the next  $\tau^\infty$ -stage, if  $x$  enters  $V$  put  $\delta(x, s)$  into both  $A$  and  $B$ , and correct  $\Delta$ .
- ▶ **Case 2** Otherwise. Put  $t_n$  into  $A_{s+1}$ . Wait till the next  $\tau^\infty$ -stage.
  1. If  $t_n$  enters  $W$ , then  $N$  is met, and we need to do nothing else. Note that  $\Delta^B$  remains correct.
  2. If  $t_n$  enters  $V$  put  $t_n$  into  $B$  and  $\xi(t_n) = t_n + 1$  (for example) into  $A$ . Pick a large fresh number  $t_{n+1} = \delta(x, s')$ . and enter cycle  $n + 1$

# Analysis

- ▶ Notice that we keep  $B \leq_{\mathcal{T}} A$  by force.
- ▶ If we pick infinitely many  $t_n$ , then we can conclude
  1.  $\sigma$  adds an infinite computable set into  $B$  and  $A$ .
  2. Nothing we add to  $A$  enters  $W$ , so (basic module)  $W = \emptyset$  (in general,  $\Gamma_{\sigma}^B = W$ ).
  3.  $\Phi_j^B(x) \uparrow$  so  $N$  is met.
- ▶ In all other cases we will succeed in meeting  $N$  after a **finite number of cycles**, and  $\Delta^B = V$  is valid, since in the case we use  $x$ , if  $x$  enters  $V$  we correct  $\Delta^B(x)$  at the next  $\tau^{\hat{\infty}}$ -stage.

## Two $\tau$ 's one $\sigma$

- ▶ Things become more complex when we consider  $\tau_0 \hat{\infty} \preceq \tau_1 \hat{\infty} \preceq \sigma$ , with  $N_j$  at  $\sigma$  as before, and  $R_j$  at  $\tau_j$ , say.
- ▶ First we consider two in their primary phases, meaning believing  $\Pi_3^0$  but being alert for  $\Sigma_3^0$ .
- ▶ It is not reasonable that  $\tau_1$  can drive  $\delta_0(z)$  to infinity on general priority grounds (i.e. for any  $z$ ), by priorities.
- ▶ But the converse is okay by general  $\mathbf{0}'''$ -grounds, and we could restart  $\tau_1$ .
- ▶ Thus at  $\sigma \times$  will (initially) have two traces  $t_n^0 = \delta_0(x, s)$  and  $t_m^1 = \delta_1(x, s) > \delta_0(x, s)$ ; and these can be chosen from e.g. separate columns of  $\omega$ .
- ▶ The **primary** goal is to get
  1. Either have  $\delta_0(y, s) > \varphi_j(y, s)$ , (for some  $y$ ) or
  2. get  $\delta_0(x, s)$  entering  $W_0$ , not  $V_0$ , after enumeration into  $A$ .

- ▶ If this never occurs, then as in the basic module,
  1.  $\delta_0(x, s) \rightarrow \infty$ ,  $\varphi_j(x, s) \rightarrow \infty$  and  $W_0$  is empty,
  2. A computable set is enumerated into  $A$ ,
  3. And, by the way we nest  $\delta_0$  inside of  $\delta_1$ , this also drives  $\delta_1(x, s) \rightarrow \infty$ .
- ▶ So we have been enumerating  $\delta_0(x, s) < \delta_1(x, s)$  which can be both taken as  $t_n$  into  $A$  at  $\sigma$ -stages.
- ▶ We might as well assume that  $\delta_0(x, s) \not\approx \varphi_j(x, s)$  as this case is easy (more or less).
- ▶ We hit  $\tau_0$  at an expansion stage.
- ▶ Since this all looks like the basic module unless  $t_n$  enters  $W_0$ , we explore what to do when  $t_n$  enters  $W_0$ .

If  $t_n = \delta_0(x, s)$  enters  $W_0$ , then currently we have no obligations to  $\Delta_0^B$ . So we could play  $\tau_0 \hat{\infty}$  and move to  $\tau_1$ .

1. If  $t_n$  entered  $W_1$ , then we are **lucky** and have met  $N$ , and need do nothing more.
2. The universe is cruel, and of course  $t_n$  entered  $V_1$ . Thus we want to correct  $\Delta_1^B = V_1$ , and would change  $B \upharpoonright \delta_1^B(x, s)$  into  $A$  at this stage  $s_1$ . To make sure that  $\Xi^A = B$  is satisfied, we would **also** have to put (e.g.)  $t_n^0 + 1 < t_n^1$  into  $A$  at  $s_1$ . **Potentially this could later change  $V_0$ .**
3. In the second case above at the next  $\tau \hat{\infty}$ -stage  $s_2$ , we would see if  $t_n^0 + 1$  entered  $W_0$  or  $V_0$ .
4. If  $V_0$ , then we would need to correct  $\Delta_0^B(x, s)$ , again and pretend the fact that " $t_n^0$  entered  $W_0$  at  $s_1$ " never happened but could correct  $\Gamma_\sigma^B(t_n^0)$ . Now we'd be back in the basic module thinking that  $\delta(x, s) \rightarrow \infty$ .
5. If  $W_0$ , we discuss next page.

- ▶ At  $s_2$   $t_n^0 + 1$  also entered  $W_0$ . Now, we are in a bit of a quandary.
  1. The  $B$ -change at  $s_1$  allows us to correct  $\Gamma^B \upharpoonright t_0 + 1$ , with no further work.
  2. The fact that we changed  $B \upharpoonright \delta_1^B(x, s)$  at  $s_1$ , means no further work is needed for  $\Delta_1^B$  at the next  $\tau_2 \hat{\infty}$ -stage.
  3. **But we can't now continue to keep moving  $\delta_0(x, s)$  for  $s > s_2$ , since  $\tau_0$  has fulfilled its obligations.**  
 Thus the plan is to detach  $\tau_0$  from  $x$ , until  $\tau_1$  looks like it fulfils its obligations.
- ▶ To wit: We would now choose a  $t_{n,1}^0 = \delta_0(t_n^0, s_2)$  large and bigger than  $\delta_0(x, s_2) = \delta_0(x, s)$  and make this more or less  $t_{n+1}^1 = \delta_1(x, s_2)$ . (Assuming this is also a  $\tau_2 \hat{\infty}$ -stage).
- ▶ Again we only attack  $N$  at  $\sigma$  at  $\sigma$ -stages where  $\ell(j, s) > \text{all current traces}$ .
- ▶ If we ever see  $\delta(t_{n,1}^0, s_2) > \varphi_j(t_{n,1}^0, s)$  we can win by enumeration of  $t_{n,1}^0$  into  $A$  (as in the basic module, with the role of  $x$  taken by  $t_{n,1}^0$ ) and correct the  $\Delta^B$ 's.
- ▶ Assuming not, we continue until the next  $W_0$  change at a  $\tau_0 \hat{\infty}$  stage, and then work as above with the new numbers.

1. If also a  $W_1$  change then we are done.
  2. If a  $V_1$  change then we use the two step process to first correct  $\Delta_1^B(x, s)$  and then at the next  $\tau_0 \hat{\infty}$  stage, see if another  $W_0$  enumeration occurred. In this case we detach again and if not we continue.
- ▶ The **only** other possibility is that at some stage  $t$ , we see  $\varphi_j(t_n^0, t) < \delta_1(t_n^0, t)$ .
  - ▶ **Now the problem** is that inevitably  $\delta_0(t_n^0, s_2 + 1) = \delta_0(t_n^0, t) = t_n^0[s_2 + 1] < \varphi_j(t_n^0, s)$ . That is, we can correct  $\Delta_1$  if  $t_n^0$  entered  $V$ , but **not**  $\Delta_0$ .
  - ▶ It is now that we add  $t_n^0 = \delta_0(x, t)$  into  $A$ .
    1. At the next  $\tau_0 \hat{\infty}$  stage we see if  $t_n^0$  enters  $W_0$ .
    2. If it does, then we can put  $\delta_1(t_n^0, t)$  into  $A$  and  $B$ , meeting  $N$  and allowing for correction, where necessary, at the next  $\tau_i \hat{\infty}$ -stages.
    3. If it enters  $V_0$  then we would correct  $\Delta_0$  by putting  $t_n^0 + 1$  into  $A$  and  $t_n^0$  into  $B$ , and go back to the primary sequence picking  $t_{n+1}^0$ .



# Analysis

- ▶ If for any cycle we never get to a  $W_0$  change, then cycle  $i$ , based on  $t_{n,i}^0$ , gives a proof that  $W_0$  is computable, and  $\varphi_j(t_m^0, s)$  is unbounded for some  $m$ . (Outcome of kind  $(g, i)$ .)
- ▶ If there are infinitely many complete cycles resulting in a  $V_0$  change, we get a proof that  $\varphi_j(x, s)$  is unbounded, and  $W_0$  is  $B$ -computable. (Outcome  $(g, u)$ .)
- ▶ Otherwise we will win  $N$  with finite effect, and  $\Delta_0$  will be correct.
- ▶ Notice that on the assumption that  $\Delta^B = V_0$  we only need concern  $\Delta_1^B$  with  $W_1 \cap W_0$  and  $V_1 \cap W_0$ , since  $A = (W_0 \cap W_1) \sqcup (W_0 \cap V_1) \sqcup (V_0 \cap V_1) \sqcup (V_0 \cap W_1)$ . And  $V_0 = (V_0 \cap V_1) \sqcup (V_0 \cap W_1)$  meaning that  $(V_0 \cap V_1) \leq_T V_0$  and  $(V_0 \cap W_1) \leq_T V_0$ . So, it **only** when things enter  $W_0$  we even need to monitor  $\Delta_1$ .
- ▶ If either of the first two outcomes occur then  $W_0 \leq_T B$ , via  $\Gamma_\sigma^B$  and a version of  $\tau_1$  guessing this outcome will be below some kind of outcome of  $\sigma$  like  $\sigma \hat{=} g$ . It will accordingly only care about numbers entering  $W_0$  for its primary  $\Delta_1$ .

## The other configurations

- ▶ Now we have  $\tau_0 \hat{\infty} \preceq \sigma \hat{(\tau_0, g)} \prec \tau_2$ .
- ▶ This  $\tau_2$  mother “knows” that  $W_0 \leq_T B$  is proven at  $\sigma$  and an infinite stream of  $\delta_0(z, s)$ 's will be entering  $B$  and  $A$ . First suppose  $g = (g, i)$ , say.
- ▶ It only issues axioms claiming  $V_1 \cap W_0 \leq_T B$  via some  $\Delta_{\tau_2}^B$ .
- ▶ Some  $\sigma'$  extending  $\tau_2 \hat{\infty}$  has a follower  $x'$  with trace  $\delta_2(x', s) > x_\sigma$ .
- ▶ On realization via  $\sigma'$ -correct computations, bigger than  $\delta_1(x', s)$ , we can put  $\delta_1(x', s)$  into  $A$  instead of  $t_{m,i}^0$  as the case might be.
- ▶ Thus we can put  $\delta_1(x', s)$  into  $A$ .
- ▶ Since this will enter  $V_0$ , by  $\tau_0$ 's assumption,  $\Delta_1$  will be correct. If it enters  $W_0$  we meet  $\sigma$ .
- ▶ Entering  $V_0$  means that  $\tau_0$  will put (more or less) it into  $B$  to correct  $V_0$ , in which case we can move  $\delta(x', s)$  to the current  $t_{m,i}^0$ .
- ▶ On reaching  $\tau_2$  we can correct  $\Delta_1^B$  if necessary.

- ▶ Now consider a version of  $\tau_2$  below  $g = (g, u)$ , so actually lots of numbers enter  $W_0$ , but later we put correctors into  $B$  and the primary  $t_n^0$  sequence is resurrected.
- ▶ This would happen if we had  $\tau^\infty \prec \hat{\tau}_2^\infty \prec \sigma^\infty(g, u) \prec \tau_2^\infty$  where  $\hat{\tau}_2$  is the original  $\tau_2$  mother, guessing  $\Pi_3^0$ .
- ▶ The only difference is there are infinitely many  $W_0$  then  $V_0$  changes, and the inner cycle slowly goes to  $\infty$ .
- ▶ If  $\tau_2$  guesses this, when we hit  $\tau_2$  we would correct  $\Gamma_1, \Delta_2$  as appropriate since  $\tau_1$  will use  $\tau_2$ 's numbers.

## One $\tau$ , two $\sigma$ 's

- ▶ Now suppose that we have  $\tau \hat{\infty} \prec \sigma_1 \prec \sigma_2$ .
- ▶ In the case that  $\sigma_2$  extends  $\sigma_1 \hat{f}$  no problem; finite injury.
- ▶ Thus suppose that  $\sigma_2$  extends  $\sigma_1 \hat{g}$  for one of the infinitary outcomes of  $\sigma_1$  giving the  $\Sigma_3^0$  outcome for  $\tau$ .
- ▶ Thus  $\sigma_2$  expects an infinite stream of  $t_n^0$  of some type to enter  $V_0$ .
- ▶ Hence it should have **no obligations to  $\tau$  if this really is the case, but maybe it's not.** This version of  $N$  at  $\sigma_2$  believes that  $W_0 \leq_{\tau} B$  via  $\Gamma_{\sigma_1}^B$ .
- ▶ The idea is that numbers associated with  $\sigma_1$  will be shared by  $\sigma_2$  in their uses  $\Delta = \Delta_{\tau}$ .

- ▶ When we visit  $\sigma_2$  we give it some follower  $x'$ , and we will give this the current  $t_n^0$  for the current  $x$  (or  $t_{i,m}^0$ ) at  $\sigma$ , for its  $\delta(x', s)$ . Note that if this is on  $TP$  then this use will be driven to  $\infty$  by  $\sigma_1$ , **but that's okay**.
- ▶ We don't believe that the computation at  $\sigma_2$  is correct unless  $\ell(\sigma_2, s) > x'$  via  $\sigma_2$ -correct computations. (After all, an infinite stream is entering  $B$  at  $\sigma_1$ .)
- ▶ Put  $x'$  into  $A$ .
- ▶ At the next  $\tau^{\hat{\infty}}$  stage  $s_1$  after  $s$ , At the next  $\tau^{\hat{\infty}}$ -stage  $t$  see which or  $W$  or  $V$   $x'$  enters.
  1. If  $W$ , **then  $\sigma_1$  is met**.
  2. If  $V$  then put  $t_n^0$  into both  $A$  and  $B$ .
  3. At the next  $\tau^{\hat{\infty}}$ -stage if  $t_n^0$  enters  $W$  make  $\Gamma_{\sigma_1}^B(t_n^0) = 1$  else  $\Delta^B(t_n^0) = 1$ , and in either case pick a new  $t_{n+1}^0$ .

- ▶ The rest is putting this on a a priority tree and using induction.
- ▶ This argument uses “capricious destruction” and is something that a young computability theorist should know.

Thank You