Priority arguments in descriptive set theory

Andrew Marks (UCLA), joint with Adam Day

July 28, 2020

Summary

- The decomposability conjecture is true, assuming Π^1_2 determinacy. This conjecture characterizes which Borel functions are piecewise continuous.

Summary

- The decomposability conjecture is true, assuming Π^1_2 determinacy. This conjecture characterizes which Borel functions are piecewise continuous.
- The proof uses priority arguments in the setting of descriptive set theory. They are carried out using Antonio Montalbán's true stages machinery.

I used to think that frameworks for conducting priority arguments had little value; they seemed ad-hoc and unable capture the ever-increasing complexity of all possible computer programs.

I used to think that frameworks for conducting priority arguments had little value; they seemed ad-hoc and unable capture the ever-increasing complexity of all possible computer programs.

> *Software engineering should be known as "The Doomed Discipline," doomed because it cannot even approach its goal since its goal is self-contradictory [...] "How to program if you cannot." –Edsger Dijkstra*

I used to think that frameworks for conducting priority arguments had little value; they seemed ad-hoc and unable capture the ever-increasing complexity of all possible computer programs.

> *Software engineering should be known as "The Doomed Discipline," doomed because it cannot even approach its goal since its goal is self-contradictory [...] "How to program if you cannot."* –*Edsger Dijkstra*

I was wrong: True stages are a robust framework which have also arisen independently outside of computability theory. True stages are the precise tool needed to deeply understand how membership in a Σ^0_n set is witnessed in boldface descriptive set theory. Montalbán's true stages machinery was essentially invented independently by Louveau and Saint-Raymond in set theory for proving Borel Wadge determinacy in second order arithmetic (Day-Greenberg-Turetsky).

The Borel hierarchy

A **Polish space** is a completely metrizable separable topological space. For example, $\mathbb{R}$, 2^ω, ω^ω, $C[0,1]$.

The Borel hierarchy

A **Polish space** is a completely metrizable separable topological space. For example, $\mathbb{R}$, 2^ω, ω^ω, $C[0,1]$.

The Borel hierarchy on a Polish space is defined as follows:

- The Σ^0_1 sets are defined to be the open sets.
- The Π^0_α sets are the complements of Σ^0_α sets.
- A set A is Σ^0_α if we can express A as a countable union $A = \bigcup_i B_i$ where each B_i is a Π^0_β set for some $\beta < \alpha$

The Borel hierarchy

A **Polish space** is a completely metrizable separable topological space. For example, $\mathbb{R}$, 2^ω, ω^ω, $C[0,1]$.

The Borel hierarchy on a Polish space is defined as follows:

- The Σ^0_1 sets are defined to be the open sets.
- The Π^0_α sets are the complements of Σ^0_α sets.
- A set A is Σ^0_α if we can express A as a countable union $A = \bigcup_i B_i$ where each B_i is a Π^0_β set for some $\beta < \alpha$

We compare the complexity of sets by **Wadge reducibility**: if $A \subseteq X$, and $B \subseteq Y$, then $A \leq_W B$ if there is a continuous function $f : X \to Y$ such that for all $x \in X$, we have $x \in A \iff f(x) \in B$.

The Borel hierarchy

A **Polish space** is a completely metrizable separable topological space. For example, $\mathbb{R}$, 2^ω, ω^ω, $C[0,1]$.

The Borel hierarchy on a Polish space is defined as follows:

- The Σ^0_1 sets are defined to be the open sets.
- The Π^0_α sets are the complements of Σ^0_α sets.
- A set A is Σ^0_α if we can express A as a countable union $A = \bigcup_i B_i$ where each B_i is a Π^0_β set for some $\beta < \alpha$

We compare the complexity of sets by **Wadge reducibility**: if $A \subseteq X$, and $B \subseteq Y$, then $A \leq_W B$ if there is a continuous function $f : X \to Y$ such that for all $x \in X$, we have $x \in A \iff f(x) \in B$.

A set B is Σ^0_α-**hard** if $A \leq_W B$ for every Σ^0_α set A. A set is Σ^0_α **complete** if it is both Σ^0_α and Σ^0_α-hard.

An easy priority argument

Assume X is Polish. $A \subseteq X$ is **meager** if its contained in a countable union of nowhere dense sets. So $B \subseteq X$ is comeager iff there are countably many dense open sets $\{D_i\}_{i \in \omega}$ so $\bigcap D_i \subseteq B$.

An easy priority argument

Assume X is Polish. $A \subseteq X$ is **meager** if its contained in a countable union of nowhere dense sets. So $B \subseteq X$ is comeager iff there are countably many dense open sets $\{D_i\}_{i \in \omega}$ so $\bigcap D_i \subseteq B$.

Say a collection $\{F_n\}_{n \in \omega}$ of closed subsets of X is **good** if $\{F_n\}_{n \in \omega}$ is closed under finite intersections, and for all F_n and all open $U \subseteq X$, if $F_n \cap U \neq \emptyset$ then there is some m such that $F_m \subseteq F_n \cap U$.

An easy priority argument

Assume X is Polish. $A \subseteq X$ is **meager** if its contained in a countable union of nowhere dense sets. So $B \subseteq X$ is comeager iff there are countably many dense open sets $\{D_i\}_{i\in\omega}$ so $\bigcap D_i \subseteq B$.

Say a collection $\{F_n\}_{n\in\omega}$ of closed subsets of X is **good** if $\{F_n\}_{n\in\omega}$ is closed under finite intersections, and for all F_n and all open $U \subseteq X$, if $F_n \cap U \neq \emptyset$ then there is some m such that $F_m \subseteq F_n \cap U$.

Theorem (Abstraction of a folklore technique)

$A \subseteq X$ is Σ^0_3-hard iff there are good closed sets $\{F_n\}_{n\in\omega}$ in X so:

- *If $x \in X$ is an element of infinitely many F_n, then $x \notin A$.*
- *For all F_n, A is comeager in F_n.*

An easy priority argument

Assume X is Polish. $A \subseteq X$ is **meager** if its contained in a countable union of nowhere dense sets. So $B \subseteq X$ is comeager iff there are countably many dense open sets $\{D_i\}_{i\in\omega}$ so $\bigcap D_i \subseteq B$.

Say a collection $\{F_n\}_{n\in\omega}$ of closed subsets of X is **good** if $\{F_n\}_{n\in\omega}$ is closed under finite intersections, and for all F_n and all open $U \subseteq X$, if $F_n \cap U \neq \emptyset$ then there is some m such that $F_m \subseteq F_n \cap U$.

Theorem (Abstraction of a folklore technique)

$A \subseteq X$ is Σ^0_3-hard iff there are good closed sets $\{F_n\}_{n\in\omega}$ in X so:

- *If $x \in X$ is an element of infinitely many F_n, then $x \notin A$.*
- *For all F_n, A is comeager in F_n.*

Proof sketch: $\Leftarrow$: This is a finite-injury priority argument. Let $C = \{x \in 2^\omega : (\exists i)(\exists^\infty j)\, x(\langle i,j\rangle) = 1\}$ be the set of reals with some infinite column. This is a complete Σ^0_3 set in the Borel hierarchy. It suffices to construct a continuous reduction from C to A. To simplify notation, assume $X = 2^\omega$.

Proof sketch of $\Leftarrow$:

For each closed F_n, let $\{D_{m,n}\}_{m\in\omega}$ be open dense subsets of F_n so that $\bigcap_m D_{m,n} \subseteq A$.
To each string $s \in 2^{<\omega}$, we'll associate a string $\rho(s) \in 2^{<\omega}$ and a list $\psi(s) = (F_0^s, \ldots, F_{k_s}^s)$ of decreasing closed sets so $\rho(s) \geq |s|$ and

$$[\rho(s)] \cap \bigcap_{k \leq k_s} F_k^s \neq \emptyset \qquad (*)$$

Proof sketch of $\Leftarrow$:

For each closed F_n, let $\{D_{m,n}\}_{m\in\omega}$ be open dense subsets of F_n so that $\bigcap_m D_{m,n} \subseteq A$.
To each string $s \in 2^{<\omega}$, we'll associate a string $\rho(s) \in 2^{<\omega}$ and a list $\psi(s) = (F_0^s, \dots, F_{k_s}^s)$ of decreasing closed sets so $\rho(s) \geq |s|$ and

$$[\rho(s)] \cap \bigcap_{k\leq k_s} F_k^s \neq \emptyset \tag{*}$$

For the empty string $\emptyset$, let $\rho(\emptyset) = \emptyset$, and $\psi(\emptyset) = ()$. Define $\rho(s)$ and $\psi(s)$ recursively as follows, where $|s| = \langle i, j\rangle$:

- (For $s^\frown 0$, extend our list of closed sets). Define $\psi(s^\frown 0)$ extending $\psi(s)$ so $|\psi(s^\frown 0)| \geq i$ and $\rho(s^\frown 0) \supseteq \rho(s)$ so that (*) holds for $s^\frown 0$.
- (For $s^\frown 1$, injure closed sets beyond F_i^s and meet next dense open sets $D_{m,i}$ in F_i^s). Define $\psi(s^\frown 1) = \psi(s) \upharpoonright i$. Let $\rho(s^\frown 1)$ be such that $[\rho(s^\frown 1)] \subseteq D_{m,i}$ for all $m \leq j$ and (*) holds.

Define the reduction f from C to A by $f(x) = \bigcup_n \rho(x \upharpoonright n)$.

Proof sketch of $\Rightarrow$:

$\Rightarrow$: First check that there are such closed sets $\{F_n\}_{n\in\omega}$ witnessing the theorem for C. If A is Σ^0_3 hard, there is a continuous reduction of C to A. By a lemma of Harrington (in Steel (1980) "Analytic sets and Borel isomorphisms"), there is a injective continuous reduction g of C to A. Take image of the sets $\{F_n\}_{n\in\omega}$ under g. $\square$

Generalizing this theorem throughout the Borel hierarchy

If $\mathcal{A}$ is a countable collection of subsets of X, let $\tau(\mathcal{A})$ denote the topology generated by the subbasis $\mathcal{A}$.

Generalizing this theorem throughout the Borel hierarchy

If $\mathcal{A}$ is a countable collection of subsets of X, let $\tau(\mathcal{A})$ denote the topology generated by the subbasis $\mathcal{A}$.

Given Polish X, say that $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ is a **suitable sequence of length** $n+1$ iff $\mathcal{A}_0$ is a countable basis of open sets for X, $\mathcal{A}_m$ is a countable set of Π^0_m subsets of X for $m \geq 1$, every $\mathcal{A}_m$ is closed under finite intersections, and for all $m < n$,

1. If $B \in \mathcal{A}_0$, then $\overline{B} \in \mathcal{A}_1$, and $\mathcal{A}_m \subseteq \mathcal{A}_{m+1}$ for $m > 0$.
2. If $B \in \mathcal{A}_m$, then $X \setminus B \in \mathcal{A}_{m+1}$.
3. If $B \in \mathcal{A}_{m+1}$, then B is closed in $\tau(\mathcal{A}_m)$.
4. If $B \in \mathcal{A}_{m+1}$ and $m > 0$, then $\overline{B}^{\mathcal{A}_{m-1}} \in \mathcal{A}_m$.

Properties (1)-(3) are simple properties which ensure that the topology $\tau(\mathcal{A}_m)$ is Polish. Property (4) here is the difficult property to satisfy. It is key to the following theorem:

Characterizing Σ^0_{n+2} hardness

Theorem (Day-M.)

Suppose X is Polish, $Y \subseteq X$, and $n \geq 1$. Then Y is Σ^0_{n+2}-hard (i.e. there exists a continuous reduction of a complete Σ^0_{n+2} set to Y) if and only if there exists a closed set $F \subseteq X$ and a suitable sequence of sets $\mathcal{A}_0, \ldots, \mathcal{A}_n$ on F such that

1. *Y is $\tau(\mathcal{A}_n)$-meager*
2. *Y is $\tau(\mathcal{A}_{n-1})$-comeager in A for all $A \in \mathcal{A}_n$*

Characterizing Σ^0_{n+2} hardness

Theorem (Day-M.)

Suppose X is Polish, $Y \subseteq X$, and $n \geq 1$. Then Y is Σ^0_{n+2}-hard (i.e. there exists a continuous reduction of a complete Σ^0_{n+2} set to Y) if and only if there exists a closed set $F \subseteq X$ and a suitable sequence of sets $\mathcal{A}_0, \ldots, \mathcal{A}_n$ on F such that

1. *Y is $\tau(\mathcal{A}_n)$-meager*
2. *Y is $\tau(\mathcal{A}_{n-1})$-comeager in A for all $A \in \mathcal{A}_n$*

The proof of $\Leftarrow$ heavily uses the true stages machinery. We construct a continuous reduction in stages. To each finite string s, we associate an approximation to $f(x)$ which consists of a sequence $(A_0, A_1, \ldots, A_n)$ of sets with nonempty intersection where $A_i \in \mathcal{A}_i$. True stages control the flow of the construction; we will have $f(x) \in A_i$ if $|s|$ is an i-true stage:

Lemma (Montalbán 2014, relativized version)

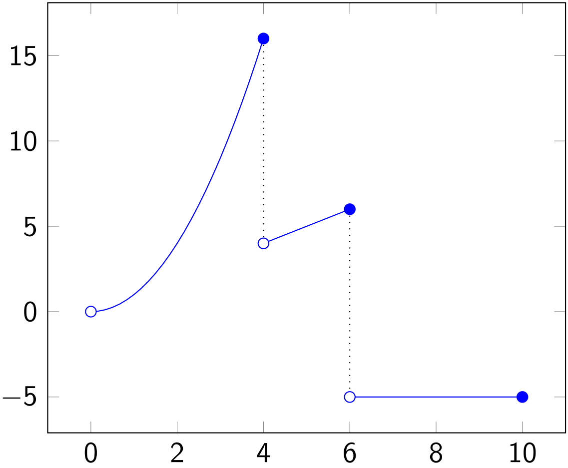
There are partial orders $\{\leq_k\}_{k\in\omega}$ *on* $2^{<\omega}$ *and a set* $S_k \subseteq 2^{<\omega}$ *such that:*

1. $\leq_0$ *is the usual prefix ordering:* $\sigma \leq_0 \tau$ *iff* $\sigma \subseteq \tau$.
2. *The empty sequence* $\emptyset$ *has* $\emptyset \leq_k \sigma$ *for every* $\sigma \in T$.
3. *If* $\sigma \leq_{k+1} \tau$, *then* $\sigma \leq_k \tau$.
4. *If* $\sigma \leq_k \tau$ *and* $\sigma \in S_k$, *then* $\tau \in S_k$.

(♣) *If* $\sigma \subseteq \tau \subseteq \rho$ *and* $\sigma \leq_{k+1} \rho$ *and* $\tau \leq_k \rho$, *then* $\sigma \leq_{k+1} \tau$.

Let T_k *be the tree* $T_k = \{(\sigma_0, \ldots, \sigma_m)\colon \sigma_i <_k \sigma_{i+1} \wedge \forall i < m\,(\neg\exists\tau(\sigma_i <_k \tau <_k \sigma_{i+1}))\}$ *of increasing* $<_k$ *sequences. Then for each* $x \in 2^\omega$ *the restriction of* T_k *to* x*:* $T_k \restriction \{(\sigma_0, \ldots, \sigma_m)\colon (\forall i)\sigma_i \subseteq x\}$ *has a single infinite branch, and we say* $\tau \subseteq x$ *is a* k-true stage of x *if* τ *is an element in this unique infinite branch. Then* $\{x\colon S_k$ *meets the* k*-true stages of* $x\}$ *is* Σ^0_{k+1} *complete.*

An application: what functions are piecewise continuous?



The decomposability conjecture

Suppose $f\colon X \to Y$ is a piecewise continuous function where $f = \bigcup_{i\in\omega} f_i$, where the f_i are partial continuous functions with Δ^0_n domains.

The decomposability conjecture

Suppose $f\colon X \to Y$ is a piecewise continuous function where $f = \bigcup_{i\in\omega} f_i$, where the f_i are partial continuous functions with Δ^0_n domains.

Then if A is is a Σ^0_n set, $f_i^{-1}(A)$ is relatively Σ^0_n in $\mathrm{dom}(f_i)$ (which is Δ^0_n), so it is Σ^0_n. Thus, $f^{-1}(A) = \bigcup_i f_i^{-1}(A)$ is a countable union of Σ^0_n sets, and hence is Σ^0_n.

The decomposability conjecture

Suppose $f\colon X \to Y$ is a piecewise continuous function where $f = \bigcup_{i\in\omega} f_i$, where the f_i are partial continuous functions with Δ^0_n domains.

Then if A is is a Σ^0_n set, $f_i^{-1}(A)$ is relatively Σ^0_n in $\mathrm{dom}(f_i)$ (which is Δ^0_n), so it is Σ^0_n. Thus, $f^{-1}(A) = \bigcup_i f_i^{-1}(A)$ is a countable union of Σ^0_n sets, and hence is Σ^0_n.

Conjecture (2000's, various authors)

$f\colon X \to Y$ is a countable union of continuous functions with Δ^0_n domains iff the preimage of every Σ^0_n set is Σ^0_n.

The decomposability conjecture

Conjecture (2000's, the decomposability conjecture)

$f : X \to Y$ is a countable union of Baire class m functions with Δ^0_n domains iff the preimage of every Σ^0_{n-m+1} set is Σ^0_n.

The decomposability conjecture

Conjecture (2000's, the decomposability conjecture)

$f : X \to Y$ is a countable union of Baire class m functions with Δ^0_n domains iff the preimage of every Σ^0_{n-m+1} set is Σ^0_n.

Theorem (Day-M.)

The decomposability conjecture is true assuming Σ^1_2 determinacy.

Prior progress

Theorem (Jayne-Rogers, 1982)

The decomposability conjecture is true for $n = 2$.

Prior progress

Theorem (Jayne-Rogers, 1982)

The decomposability conjecture is true for $n = 2$.

Theorem (Ding-Kihara-Semmes-Zhao, 2018)

The decomposability conjecture is true for $n = 3$.

Prior progress

Theorem (Jayne-Rogers, 1982)

The decomposability conjecture is true for $n = 2$.

Theorem (Ding-Kihara-Semmes-Zhao, 2018)

The decomposability conjecture is true for $n = 3$.

These theorems are proved the following way: Suppose $f : X \to Y$ is not a union of Baire class m functions with Δ^0_n domains. Then construct a Σ^0_{n-m+1} set A whose preimage is not Σ^0_n (i.e. $f^{-1}(A)$ is Π^0_n hard).

Prior progress

Theorem (Jayne-Rogers, 1982)

The decomposability conjecture is true for $n = 2$.

Theorem (Ding-Kihara-Semmes-Zhao, 2018)

The decomposability conjecture is true for $n = 3$.

These theorems are proved the following way: Suppose $f : X \to Y$ is not a union of Baire class m functions with Δ^0_n domains. Then construct a Σ^0_{n-m+1} set A whose preimage is not Σ^0_n (i.e. $f^{-1}(A)$ is Π^0_n hard).

Proposition

To prove the decomposability conjecture, it's enough to prove the case where $m = n - 1$.

Changing topology

Suppose we change our topology (X, τ) on X to a new Polish topology (X, η) where we make countably many Π^0_n sets in (X, τ) the new basic open sets of (X, η).

Changing topology

Suppose we change our topology (X, τ) on X to a new Polish topology (X, η) where we make countably many Π^0_n sets in (X, τ) the new basic open sets of (X, η).

- Any Σ^0_1 set in (X, η) is Σ^0_{n+1} in (X, τ).

Changing topology

Suppose we change our topology (X, τ) on X to a new Polish topology (X, η) where we make countably many Π^0_n sets in (X, τ) the new basic open sets of (X, η).

- Any Σ^0_1 set in (X, η) is Σ^0_{n+1} in (X, τ).
- Any Σ^0_k set in (X, η) is Σ^0_{n+k} in (X, τ).

Changing topology

Suppose we change our topology (X, τ) on X to a new Polish topology (X, η) where we make countably many Π^0_n sets in (X, τ) the new basic open sets of (X, η).

- Any Σ^0_1 set in (X, η) is Σ^0_{n+1} in (X, τ).
- Any Σ^0_k set in (X, η) is Σ^0_{n+k} in (X, τ).
- If A is not Σ^0_{n+k} in (X, τ), it is not Σ^0_k in (X, η).

Changing topology

Suppose we change our topology (X, τ) on X to a new Polish topology (X, η) where we make countably many Π^0_n sets in (X, τ) the new basic open sets of (X, η).

- Any Σ^0_1 set in (X, η) is Σ^0_{n+1} in (X, τ).
- Any Σ^0_k set in (X, η) is Σ^0_{n+k} in (X, τ).
- If A is not Σ^0_{n+k} in (X, τ), it is not Σ^0_k in (X, η).

The converses of these statements are **very** false.

A naive attempt at proving decomposability

- Suppose $f\colon (X, \tau) \to Y$ is not a union of Baire class $n-1$ functions with Δ^0_n domains.

A naive attempt at proving decomposability

- Suppose $f\colon (X,\tau) \to Y$ is not a union of Baire class $n-1$ functions with Δ^0_n domains.
- Change the topology (X,τ) to (X,η) by making some Π^0_{n-2} sets in (X,τ) the new basic open sets of (X,η).

A naive attempt at proving decomposability

- Suppose $f\colon (X,\tau) \to Y$ is not a union of Baire class $n-1$ functions with Δ^0_n domains.
- Change the topology (X,τ) to (X,η) by making some Π^0_{n-2} sets in (X,τ) the new basic open sets of (X,η).
- Then $f\colon (X,\eta) \to Y$ is not a union of Baire class 1 functions with Δ^0_3 domains.

A naive attempt at proving decomposability

- Suppose $f\colon (X,\tau) \to Y$ is not a union of Baire class $n-1$ functions with Δ^0_n domains.
- Change the topology (X,τ) to (X,η) by making some Π^0_{n-2} sets in (X,τ) the new basic open sets of (X,η).
- Then $f\colon (X,\eta) \to Y$ is not a union of Baire class 1 functions with Δ^0_3 domains.
- Apply the techniques of Ding-Kihara-Semmes-Zhao and obtain a Σ^0_2 set $A \subseteq Y$ so that $f^{-1}(A)$ is not Σ^0_3 in (X,η).

A naive attempt at proving decomposability

- Suppose $f\colon (X,\tau) \to Y$ is not a union of Baire class $n-1$ functions with Δ^0_n domains.
- Change the topology (X,τ) to (X,η) by making some Π^0_{n-2} sets in (X,τ) the new basic open sets of (X,η).
- Then $f\colon (X,\eta) \to Y$ is not a union of Baire class 1 functions with Δ^0_3 domains.
- Apply the techniques of Ding-Kihara-Semmes-Zhao and obtain a Σ^0_2 set $A \subseteq Y$ so that $f^{-1}(A)$ is not Σ^0_3 in (X,η).
- Hope that this set which is not Σ^0_3 in (X,η) is not Σ^0_n in the original topology (X,τ).

This works eventually if we just keep trying

Each time our idea fails, we get a canonical new Π^0_{n-2} set to add to our change of topology. (This requires a careful analysis of the $n = 3$, $m = 2$ decomposability proof).

This works eventually if we just keep trying

Each time our idea fails, we get a canonical new Π^0_{n-2} set to add to our change of topology. (This requires a careful analysis of the $n = 3$, $m = 2$ decomposability proof).

If we add this set to our change of topology and try again, we'll eventually succeed at some countable ordinal stage. If not, we would contradict the following:

Theorem (Harrington 1978, AD)

Fix $\alpha < \omega_1$. There is no ω_1 length sequence of distinct Π^0_α sets.

This works eventually if we just keep trying

Each time our idea fails, we get a canonical new Π^0_{n-2} set to add to our change of topology. (This requires a careful analysis of the $n = 3$, $m = 2$ decomposability proof).

If we add this set to our change of topology and try again, we'll eventually succeed at some countable ordinal stage. If not, we would contradict the following:

Theorem (Harrington 1978, AD)

Fix $\alpha < \omega_1$. There is no ω_1 length sequence of distinct Π^0_α sets.

This way of proving the theorem is cheating. A better way (which wouldn't use determinacy) would be to use standard techniques/reflection phenomena from Gandy-Harrington forcing to define from the outset the correct suitable sequence of sets needed to witness there is a Σ^0_{n-m+1} set with a preimage that is Π^0_n hard. (See e.g. Louveau's characterization of lightface Δ^1_1 sets that are boldface Σ^0_α for a construction of such suitable sequences).

This works eventually if we just keep trying

Each time our idea fails, we get a canonical new Π^0_{n-2} set to add to our change of topology. (This requires a careful analysis of the $n = 3$, $m = 2$ decomposability proof).

If we add this set to our change of topology and try again, we'll eventually succeed at some countable ordinal stage. If not, we would contradict the following:

Theorem (Harrington 1978, AD)

Fix $\alpha < \omega_1$. There is no ω_1 length sequence of distinct Π^0_α sets.

This way of proving the theorem is cheating. A better way (which wouldn't use determinacy) would be to use standard techniques/reflection phenomena from Gandy-Harrington forcing to define from the outset the correct suitable sequence of sets needed to witness there is a Σ^0_{n-m+1} set with a preimage that is Π^0_n hard. (See e.g. Louveau's characterization of lightface Δ^1_1 sets that are boldface Σ^0_α for a construction of such suitable sequences).
This is still open.

Thanks!