Randomness and ITTM

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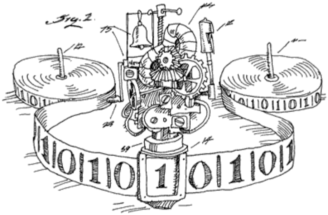


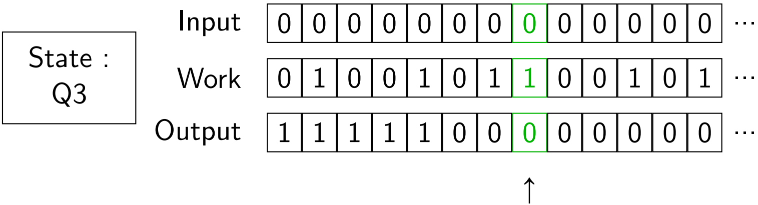
Section 1

The ITTM model

Infinite time Turing machines

An infinite-time Turing machine is a Turing machine with three tapes whose cells are indexed by natural numbers :

- The input tape
- The output tape
- The working tape



It behaves like a standard Turing machine at successor steps of computation.

Infinite time Turing machines

At limit steps of computation :

- The head goes back to the first cell.
- The machine goes into a “limit” state.
- The value of each cell equals the lim inf of the values at previous stages of computation.

State : limit

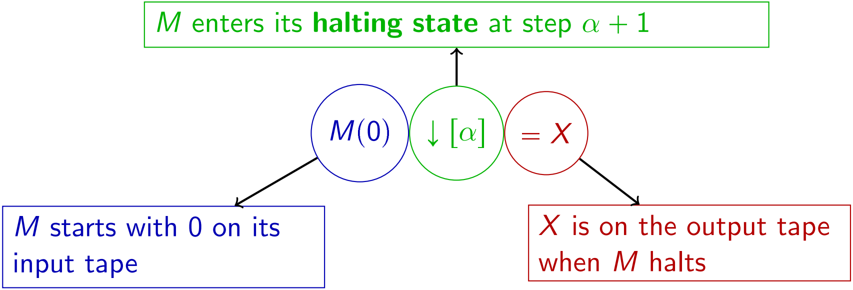
Input	0	0	0	0	0	0	0	0	0	0	0	0	0	⋯
Work	0	1	0	0	1	0	1	1	0	0	1	0	1	⋯
Output	1	1	1	1	1	0	0	0	0	0	0	0	0	⋯
	↑													

Writable reals

What is the equivalent of computable for an ITTM ?

Definition

A real X is **writable** if there in an ITTM M such that :
$M(0) \downarrow [\alpha] = X$ for some ordinal α.



Decidable classes

Which reals are writable ?

Definition

A class of real $\mathcal{A}$ is **decidable** if there is an ITTM M such that $M(X)\downarrow = 1$ if $X \in \mathcal{A}$ and $M(X)\downarrow = 0$ if $X \notin \mathcal{A}$.

Proposition (Hamkins, Lewis)

The class of reals coding for a well-order (with the code $X(\langle n, m\rangle) = 1$ iff $n < m$) is decidable.

Decide well-orders

Proposition (Hamkins, Lewis)

The class of reals coding for a well-order (with the code $X(\langle n, m\rangle) = 1$ iff $n < m$) is decidable.

The algorithm is as follow, where $<$ is the order coded by X :

Algorithm to decide well-orders

while $<$ is not empty **do**
  Look for the smallest element a of $<$ (coded by X)
  if there is no smallest element **then**
    write 0 and halts
  else
    remove a from the support of $<$
  end
end
When $<$ is empty, write 1 and halts.

Decide well-orders

How to find the smallest element ?

Algorithm to find the smallest element

```
Write 1 on the first cell. Set the current element c = +∞
if state is successor then
    if there exists a < c then
        Update c = a
        Flip the first cell to 0 and then back to 1
    end
else
    if If the first cell is 0 then
        There is no smallest element
    else
        c is the smallest element
    end
end
```

Decidable and writable sets

Proposition (Hamkins, Lewis)

The class of reals coding for a well-order (with the code $X(\langle n, m\rangle) = 1$ iff $n < m$) is decidable.

Corollary (Hamkins, Lewis)

Every Π^1_1 set of reals is decidable.

Corollary (Hamkins, Lewis)

Every Π^1_1 set of integers is writable.

Computational power of ITTM

ω_1^{ck} steps of computations are enough to write any Π_1^1 set of integers. But there is no bound in the ordinal step of computation an ITTM can use.

Using a program that writes Kleene's O, we can design a program which writes the double hyperjump O^O and then $O^{(O^O)}$ and so on.

Where does it stop ?

Proposition (Hamkins, Lewis)

Whatever an ITTM does, it does it before stage ω_1.

Computational power of ITTM

Proposition (Hamkins, Lewis)

Whatever an ITTM does, it does it before stage ω_1.

The configuration of an ITTM is given by :

1. Its tapes
2. Its state
3. The position of the head.

Let $C(\alpha) \in 2^\omega$ be a canonical encoding of the tapes of an ITTM at stage α.

There must be some *limit ordinal* $\alpha < \omega_1$ such that $C(\alpha) = C(\omega_1)$. The full configuration of the machine at step ω_1 is then the same than the one step α.

Computational power of ITTM

ω_1	0	1	0	0	0	0	0	0	0	1	0	0	1	$\cdots$
							$\dots$							
$\sup_n \alpha_n^+$	0	1	0	0	0	0	0	0	0	1	0	0	1	$\cdots$
							$\dots$							
$\alpha_2^+ > \alpha_1^+$	0	1	0	1	0	1	0	0	1	1	1	1	1	$\cdots$
$\alpha_1^+ > \alpha_0$	1	1	1	1	0	1	0	0	1	1	1	1	1	$\cdots$
α_0	0	1	1	1	0	1	0	0	1	1	1	1	1	$\cdots$

α_0 : The smallest ordinal such that every cell converging at step ω_1 (in green) will never change pass that point.

α_{n+1}^+ : The smallest ordinal $> \alpha_n^+$ such that the $n+1$ non-converging cells (in red) change value at least once in the interval $[\alpha_n^+, \alpha_{n+1}^+]$

Beyond the writable ordinals

Definition (Hamkins, Lewis)

An ordinal α is **writable** if there is an ITTM which writes an encoding of a well-order of ω with order-type α.

Proposition (Hamkins, Lewis)

The writables are all initial segments of the ordinals.

Definition (Hamkins, Lewis)

Let λ be the supremum of the writable ordinals.

Proposition (Hamkins, Lewis)

There is an ITTM which writes λ on its output tape, then leave the output tape unchanged without ever halting.

Beyond the writable ordinals

Proposition (Hamkins, Lewis)

There is a univeral ITTM U which runs simultaneously all the ITTM computations $P_e(0)$ for every $e \in \omega$.

Algorithm to eventually write λ

```
for every stage s do
    Run the universal machine U for one step.
    Compute the sum α_s of all ordinals which are on the output tapes
      of programs simulated by U[s] and which have terminated.
    Write α_s on the output tape.
end
```

Let s be the smallest stage such that every halting ITTM have halted by stage s in the simulation U.

1. We clearly have $\alpha_s \geqslant \lambda$.
2. We clearly have that $\alpha_t = \alpha_s$ for every $s \geqslant t$.

Beyond the eventually writable ordinals

Definition (Hamkins, Lewis)

A real is **eventually writable** if there in an ITTM and a step α such that for every $\beta \geqslant \alpha$, the real is on the output tape at step β.

Proposition (Hamkins, Lewis)

The eventually writable ordinals are an initial segments of the ordinals.

Definition (Hamkins, Lewis)

Let ζ be the supremum of the eventually writable ordinals.

Proposition (Hamkins, Lewis)

There is an ITTM which at some point writes ζ on its output tape.

Beyond the eventually writable ordinals

Algorithm to accidentally write ζ

for every stage s **do**
  Run the universal machine U for one step.
  Compute the sum α_s of all ordinals which are on the output tapes of programs simulated by $U[s]$.
  Write α_s on the output tape.
end

Let s be the smallest stage such that every ITTM writting an eventually writable ordinal, have done so by stage s in the simulation U. We clearly have $\alpha_s \geqslant \zeta$.

Beyond the eventually writable ordinals

Definition (Hamkins, Lewis)

A real is **accidentally writable** if there in an ITTM and a step α such that the real is on the output tape at step α.

Proposition (Hamkins, Lewis)

The accidentally writables are all initial segments of the ordinals.

Definition (Hamkins, Lewis)

Let Σ be the supremum of the accidentally writables.

Proposition (Hamkins, Lewis)

We have $\lambda < \zeta < \Sigma$.

Section 2

ITTM
and constructibility

The constructibles

Definition (Godel)

The **constructible universe** is defined by induction over the ordinals as follow :

$$\begin{aligned} L_{\varnothing} &= \varnothing \\ L_{\alpha^+} &= \{X \subseteq L_\alpha \ : \ X \text{ is f.o. definable with param. in } L_\alpha\} \\ L_{\sup_n \alpha_n} &= \bigcup_n L_{\alpha_n} \end{aligned}$$

Theorem (Hamkins, Lewis)

- If α is writable and $X \in 2^\omega \cap L_\alpha$ then X is writable.
- If α is eventually writable and $X \in 2^\omega \cap L_\alpha$ then X is eventually writable.
- If α is accidentally writable and $X \in 2^\omega \cap L_\alpha$ then X is accidentally writable.

The admissibles

Definition (Admissibility)

An ordinal α is **admissible** if L_α is a model of Σ_1-replacement. Formally for any Σ_1 formula Φ with parameters and any $N \in L_\alpha$ we must have :

$$\begin{array}{rcl} & L_\alpha & \models & \forall n \in N\ \exists z\ \Phi(n,z) \\ \rightarrow & L_\alpha & \models & \exists Z\ \forall n \in N\ \exists z \in Z\ \Phi(n,z) \end{array}$$

$\omega, \omega_1^{ck}, \omega_2^{ck}, \omega_3^{ck}, etc...$ are the first admissible ordinals.

Consider the formula $\exists n\ \forall k < n\ \exists m\ A(n,k,m)$ (with A Δ_0).
The formula is Σ_1 : This is because if for every $k < n$, there exists a witness m_k such that $A(n,k,m_k)$, then $\sup_k m_k$ is still finite.

The admissible are the sets for which this property is still true.

The admissibles

Proposition (Hamkins, Lewis)

The ordinals λ and ζ are admissible.

Suppose that for some $N \in L_\lambda$ and a Σ_1 formula Φ we have :

$$L_\lambda \models \forall n \in N \; \exists z \; \Phi(n, z)$$

We define the following ITTM :

Algorithm to show λ admissible

Write a code for N
for every $n \in N$ **do**
  Look for the first writable α_n such that $L_{\alpha_n} \models \exists z \; \Phi(n, z)$
  Write α_n somewhere.
end
Write $\sup_{n \in N} \alpha_n$

The admissibles

Proposition (Hamkins, Lewis)

The ordinals λ is the λ-th admissible.
The ordinals ζ is the ζ-th admissible.

Suppose λ is the α-th admissible for $\alpha < \lambda$.

Algorithm to show λ is the λ-th admissible

Write α
while $\alpha > 0$ **do**
    Look for the smallest element *e* of α and remove it from α
    Look for the next admissible writable ordinal and write it to the *e*-th tape
end
Write the smallest admissible greater than all the one written previously.

How big is λ

Definition

An ordinal is **recursively inaccessible** if it is admissible and limit of admissible.

Proposition (Hamkins, Lewis)

The ordinals λ is the λ-th recursively admissible.
The ordinals ζ is the ζ-th recursively admissible.

Definition

An ordinal is **meta-recursively inaccessible** if it is admissible and a limit of recursively inaccessible.

Proposition (Hamkins, Lewis)

The ordinals λ is the λ-th meta recursively admissible.
The ordinals ζ is the ζ-th meta recursively admissible.

Understanding λ, ζ, Σ

Lemma (Welch)

Let $i \in \omega$. If the sequence $\{C_i(\alpha)\}_{\alpha<\lambda}$ converges, then for every $\alpha \in [\lambda, \Sigma]$ we have $C_i(\alpha) = C_i(\lambda)$.

Suppose w.l.o.g. that $\{C_i(\alpha)\}_{\alpha<\lambda}$ converges to 0.
Let β be the smallest such that for all $\alpha \in [\beta, \lambda]$ we have $C_i(\alpha) = 0$.

Algorithm

```
for every α > β written by U do
    Simulate another run of U for α steps
    if C_i(γ) = 1 for some γ ∈ [β, α] then
        Write α and halt.
    end
end
```

Suppose there is an accidentally writable ordinal $\alpha > \beta$ s.t. $C_i(\alpha) = 1$. Then U will write such an ordinal at some point, and the above program will then write $\alpha > \lambda$ and halt. This is a contradiction.

Understanding λ, ζ, Σ

Theorem (Welch)

The whole state of an ITTM at step ζ is the same than its state at step Σ. In particular, it enters an infinite loop at stage ζ.

The theorem follows from the two following lemmas :

Lemma (Welch)

Let $i \in \omega$. If the sequence $\{C_i(\alpha)\}_{\alpha<\zeta}$ converges, then for every $\alpha \in [\zeta, \Sigma]$ we have $C_i(\alpha) = C_i(\zeta)$.

Lemma (Welch)

Let $i \in \omega$. If the sequence $\{C_i(\alpha)\}_{\alpha<\zeta}$ diverges, then the sequence $\{C_i(\alpha)\}_{\alpha<\Sigma}$ diverges.

Understanding λ, ζ, Σ

Suppose w.l.o.g. that $\{C_i(\alpha)\}_{\alpha<\zeta}$ converges to 0.
Let β be the smallest such that for all $\alpha \in [\beta, \zeta]$ we have $C_i(\alpha) = 0$.
The ordinal β is eventually writable through different versions $\{\beta_s\}_{s \in ORD}$

Algorithm

for every s and every $\alpha > \beta_s$ written by U **do**
  Simulate another run of U for α steps
  if $C_i(\gamma) = 1$ for $\gamma \in [\beta_s, \alpha]$ and β_s has changed since **then**
    Write α on the output tape.
  end
end

Suppose there is an accidentally writable ordinal $\alpha > \beta$ s.t. $C_i(\alpha) = 1$. Then some ordinal $\alpha' \geqslant \alpha$ will be written at some stage at which β_s has stabilized. Thus the above program will then eventually write some $\alpha' > \zeta$. This is a contradiction.

Understanding λ, ζ, Σ

Suppose $\{C_i(\alpha)\}_{\alpha<\Sigma}$ converges.

Algorithm

```
Set β = 0
for every α > β written by U do
    Simulate another run of U for α steps
    if C_i(γ) changes for γ ∈ [β, α] then
        Let β = α
        Write α
    end
end
```

The algorithm will eventually write some ordinal α s.t. $\{C_i(\gamma)\}$ does not change for $\gamma \in [\alpha, \Sigma]$. But then α is eventually writable and $\{C_i(\alpha)\}_{\alpha<\zeta}$ converges.

Understanding λ, ζ, Σ

Theorem (Welch)

The whole state of an ITTM at step ζ is the same than its state at step Σ. In particular, it enters an infinite loop at stage ζ.

Corollary (Welch)

λ is the supremum of the ITTM's halting times.

Indeed, suppose that we have $M(0) \downarrow [\alpha]$ for some M and α accidentally writable. Then we can run $M(0)[\beta]$ for every β accidentally writable until we find one for which M halts, and then write β. Thus α must be writable.

Suppose now that $M(0) \uparrow [\Sigma]$. Then M will never halt. Thus if M halts, it halts at a writable step.

Understanding λ, ζ, Σ

Theorem (Welch)

The whole state of an ITTM at step ζ is the same than its state at step Σ. In particular, it enters an infinite loop at stage ζ.

Corollary (Welch)

- The writable reals are exactly the reals of L_λ.
- The eventually writable reals are exactly the reals of L_ζ.
- The accidentally writable reals are exatly the reals of L_Σ.

We can construct every successive configurations of a running ITTM. Also to compute a writable reals, there are less than λ steps of computation and then less than λ steps of construction. Thus every writable real is in L_λ.

The argumet is similar for ζ and Σ.

Understanding λ, ζ, Σ

Definition

Let $\alpha \leqslant \beta$. We say that L_α is ***n*-stable** in L_β and write $L_\alpha \prec_n L_\beta$ if

$$L_\alpha \models \Phi \leftrightarrow L_\beta \models \Phi$$

For every Σ_n formula Φ with parameters in L_α.

Theorem (Welch)

(λ, ζ, Σ) is the lexicographically smallest triplet such that :

$$L_\lambda \prec_1 L_\zeta \prec_2 L_\Sigma$$

Understanding λ, ζ, Σ

Theorem (Welch)

The ordinal Σ is not admissible.

To see this, we define the following function $f : \omega \to \Sigma$:

$$\begin{array}{rcl} f(0) & = & \zeta \\ f(n) & = & \text{the smallest } \alpha \text{ s.t. } C(\alpha)\restriction_n = C(\zeta)\restriction_n \end{array}$$

It is not very hard to show that we must have $\sup_n f(n) = \Sigma$

Theorem (Welch)

The ordinal Σ is a limit of admissible.

Otherwise, if α is the greatest admissible smaller than Σ, one could compute $\Sigma \leqslant \omega_1^{\alpha}$.

Section 4

ITTM and randomness

ITTM and randomness

Definition (Carl, Schlicht)

X is α-**random** if X is in no set whose Borel code is in L_α.

Definition

An open set U is α-**c.e.** if $U = \bigcup_{\sigma \in A} [\sigma]$ for a set $A \subseteq 2^{<\omega}$ such that :

$$\sigma \in A \leftrightarrow L_\alpha \models \Phi(\sigma)$$

for some Σ_1 formula Φ with parameters in L_α.

Definition (Carl, Schlicht)

X is α-**ML-random** if X is in no set uniform intersection $\bigcap_n U_n$ of α-c.e. open set, with $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$.

Projectibles and ML-randomness

Definition

We say that α is **projectible** into $\beta < \alpha$ if there is an injective function $f : \alpha \to \beta$ that is Σ_1-definable in L_α.
The least β such that α is projectible into β is called the **projectum** of α and denoted by α^*.

Theorem (Angles d'Auriac, Monin)

The following are equivalent for α limit such that $L_\alpha \models$ everything is countable :

- α is projectible into ω.
- There is a universal α-ML-test.
- α-ML-randomness is strictly stronger than α-randomness.

λ-ML-randomness

Theorem

The ordinal λ is projectible into ω without using any parameters.

Each writable ordinal can be effectively assigned to the code of the ITTM writting it.

Corollary

Most work in Δ^1_1 and Π^1_1-ML-randomness still work with λ-ML-randomness and λ-randomness. In particular λ-ML-randomness is strictly stronger than λ-randomness.

ζ-ML-randomness

Theorem

The ordinal ζ is not projectible into ω.

Suppose that an eventually writable parameter α can be used to have a projuctum $f : \zeta \to \omega$. Then every eventually writable ordinal become writable using α. Then ζ becomes eventually writable using α. But then ζ is eventually writable.

Corollary

ζ-randomness coincides with ζ-ML-randomness. An analogue of Ω for ζ-randomness does not exists.

Σ-ML-randomness

Theorem

The ordinal Σ is projectible into ω, using ζ as a parameter.

We can use the fact that (ζ, Σ) is the least pair such that : $C(\zeta) = C(\Sigma)$, with the function :

$$\begin{aligned} f(0) &= \zeta \\ f(n) &= \text{the smallest } \alpha \text{ s.t. } C(\alpha)\restriction_n = C(\zeta)\restriction_n \end{aligned}$$

Every ordinal $f(n)$ is then Σ_1-definable with ζ as a parameter.
As $L_\Sigma \models$ "everything is countable", it follows that every ordinal smaller than $f(n)$ for some n is Σ_1-definable with ζ as a parameter.
As $\sup_n f(n) = \Sigma$, it follows that every accidentally writable is Σ_1-definable with ζ as a parameter.
The projectum is then a code for the formula defining each ordinal.

ITTM-randomness

Definition (Hamkins, Lewis)

A class of real $\mathcal{A}$ is **semi-decidable** if there is an ITTM M such that $M(X)\downarrow$ if $X \in \mathcal{A}$.

Definition (Carl, Schlicht)

A sequence X is **ITTM-random** if X is in no semi-decidable set of measure 0.

Definition

We say that X is low for λ if $\lambda^X = \lambda$.
We say that X is low for ζ if $\zeta^X = \zeta$.
We say that X is low for Σ if $\Sigma^X = \Sigma$.

ITTM-randomness

Theorem (Carl, Schlicht)

The following are equivalent for a sequence X :

1. X is ITTM-random
2. X is Σ-random and $\Sigma^X = \Sigma$
3. X is ζ-random and $\Sigma^X = \Sigma$

Theorem (M., Angles d'Auriac)

We have :

1. λ-randoms $\subsetneq$ λ-ML-randoms $\subsetneq$ ITTM-randoms
2. ζ-randoms $=$ ζ-ML-randoms $\subsetneq$ ITTM-randoms
3. Σ-randoms $\subseteq$ ITTM-randoms $\subsetneq$ Σ-ML-randoms

ITTM-randomness

Question

Does there exists X such that X is Σ-random but not ITTM random ?

→ Does there exists a Σ-random set X such that $L_\zeta[X] \not\prec_2 L_\Sigma[X]$?

Fact : If X is Σ-random we have $L_\lambda[X] \prec_1 L_\zeta[X] \prec_1 L_\Sigma[X]$

Presumably easier question

Does there exists any set X such that $L_\zeta[X] \prec_1 L_\Sigma[X]$ but $L_\zeta[X] \not\prec_2 L_\Sigma[X]$?

ITTM-randomness

Question

Does there exists any set $X \subseteq \omega$ such that $L_\zeta[X] \prec_1 L_\Sigma[X]$ but $L_\zeta[X] \not\prec_2 L_\Sigma[X]$?

We would need a set X which encodes Σ, but with a decoding requiring at least Σ steps to be performed...
Proof sketch by Sy Friedman :

1. Perform ζ-Cohen forcing to build a subset $A \subseteq \zeta$ such that for any extension $B \geq A$ with $B \subseteq \Sigma$ we have $L_\zeta[B] \prec_1 L_\Sigma[B]$.
2. Consider the extension B of A which adds only 0 on ordinals bigger than ζ
3. Note that we must have $L_\zeta[B] \not\prec_2 L_\Sigma[B]$ as B is co-final below ζ but not below Σ
4. Perform almost disjoint forcing to find a set $x \subseteq \omega$ which is able to encode B in Σ steps and such that $L_\zeta[B][x] \prec_1 L_\Sigma[B][x]$

Almost disjoint coding

Given a set $A \subseteq \Sigma$ consider Σ-definable almost disjoint sets $\{X_\alpha\}_{\alpha<\Sigma}$. The forcing conditions are partial functions $p : \omega \to \{0,1\}$ such that :

- dom $p \cap X_\alpha$ is finite for $\alpha \in A$
- $\{n \ : \ p(n) = 1\}$ is finite.

A sufficiently generic set $G \subseteq \omega$ will be such that $G \cap X_\alpha$ is infinite iff $\alpha \in A$.
$\to L_\Sigma[A][G] = L_\Sigma[G]$ and $L_\zeta[A][G] = L_\zeta[G]$.

We can the use the real G to decode $A \subseteq \Sigma$ but so to speak only one ordinal at a time, when they appear. In order to do so we need to make sure each X_α is Cohen-generic over L_α relative to $\{X_\gamma\}_{\gamma<\alpha}$.

Almost disjoint coding

Almost disjoint coding was used by Beller, Jensen and Welch in a 300 pages proof called "coding the universe" to show that any model M of ZFC is a submodel of $L[X]$ for some $X \subseteq \omega$.

The idea is :

1. First to make M a submodel of $L[A]$ for $A \subseteq ORD$ a proper class of ordinals (done by Lévy).
2. Then to perform iterated almost disjoint coding to be able to retrieve more and more of A as more and more ordinals become available. This step is decomposed cardinal by cardinal.

Another question raised

As we play with stability, this leads to another question.

Theorem (Sacks)

A countable ordinal α is admissible iff $\alpha = \omega_1^X$ for some X.

Can we obtain a similar result with ITTM ?

Question

Let α, β, γ be countable ordinals such that $L_\alpha <_1 L_\beta <_2 L_\gamma$. Do we have $X \subseteq \omega$ such that $\alpha = \lambda^X, \beta = \zeta^X$ and $\gamma = \Sigma^X$?

Conjecture : yes with some forcing involving almost disjoint coding.

Some positive result with ITTM-genericity

Definition

A co-meager set is a countable intersection of dense open sets. The complement of a co-meager set is a meager set.

Definition

We say that X is generic over L_α if X is in every dense open set with code in L_α.

Definition

We say that X is ITTM-generic if X is in no ITTM-semi-decidable meager set.

Genericity

The theorem relating ITTM-genericity and genericity over L_Σ holds like for ITTM-randomness :

Theorem

Let X be a real. Then X is ITTM-generic $\leftrightarrow$ X is generic over L_Σ and $\Sigma^X = \Sigma$

But in fact

Theorem

If G is generic over L_Σ then $L_\zeta[G] <_2 L_\Sigma[G]$. In particular $\Sigma^G = \Sigma$.

Corollary

ITTM-genericity and genericity over L_Σ are two equivalent notions.

Questions summary

Question

Let α, β, γ be countable ordinals such that $L_\alpha \prec_1 L_\beta \prec_2 L_\gamma$. Do we have $X \subseteq \omega$ such that $\alpha = \lambda^X, \beta = \zeta^X$ and $\gamma = \Sigma^X$?

Question

Does there exists a Σ-random set X such that $L_\zeta[X] \not\prec_2 L_\Sigma[X]$?