

Computing descending sequences in linear orderings

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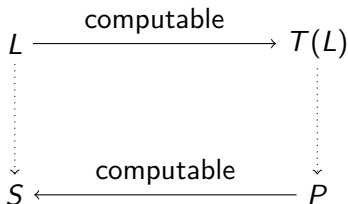
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How hard is it to compute an infinite descending sequence in an ill-founded linear ordering L ?

This problem reduces to the problem of computing a path on an ill-founded subtree of $\mathbb{N}^{<\mathbb{N}}$:

- Given L , compute the tree $T(L)$ of all finite descending sequences in L ;
- Every path P on $T(L)$ computes an infinite descending sequence S in L .

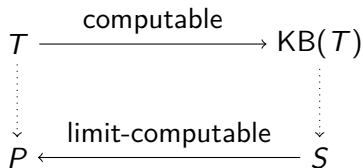


Going in the opposite direction

- Given a tree T , we can compute its **Kleene-Brouwer ordering** $\text{KB}(T)$, defined by $\sigma \leq_{\text{KB}} \tau$ iff σ extends τ or is lexicographically below τ .
- $\text{KB}(T)$ is ill-founded if and only if T is ill-founded.
- Given a descending sequence $(\sigma_i)_{i \in \mathbb{N}}$ in $\text{KB}(T)$,

$$P(n) = \lim_{i \rightarrow \infty} \sigma_i(n)$$

is a path on T .



Can we do better?

If each object in a space X can be “encoded” as a real, then we can make it into a **represented space**, thereby transferring notions of computability from $\mathbb{N}^{\mathbb{N}}$ to X .

Formally, a represented space is a pair (X, δ) where $\delta : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow X$ is a (possibly partial) surjection.

Each element of X is **named** by some (possibly multiple) $p \in \mathbb{N}^{\mathbb{N}}$ via δ .

Examples:

$$\mathbb{N}^{\mathbb{N}}, \quad \mathbb{N}, \quad \text{LO}, \quad \text{Tr}, \quad \Pi_1^1(\mathbb{N}), \quad \Sigma_1^1(\text{LO})$$

All of the above spaces can be represented in standard ways.

Examples of problems

DS: given an ill-founded linear ordering, produce any infinite descending sequence

$C_{\mathbb{N}^{\mathbb{N}}}$: given an ill-founded subtree of $\mathbb{N}^{<\mathbb{N}}$, produce any path

lim: given a convergent sequence of reals, produce its limit

Formally, a **problem** $f : \subseteq X \rightrightarrows Y$ is a (possibly partial) multivalued function between represented spaces.

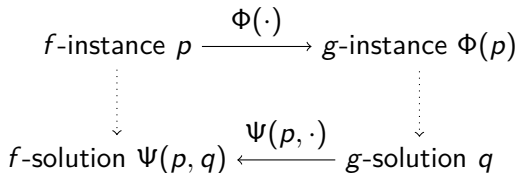
We also think of a problem as a set of instance-solution pairs:

- If $x \in \text{dom}(f)$ then we say that x is an **f -instance**.
- For each f -instance x , the set of **f -solutions** to x is $f(x) \subseteq Y$.

Definition

A problem f is **Weihrauch reducible** to a problem g ($f \leq_W g$) if there are computable functions $\Phi, \Psi : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ such that:

- if p is a name for an f -instance, then $\Phi(p)$ is a name for a g -instance;
- if p is a name for an f -instance and q is a name for a g -solution to $\Phi(p)$, then $\Psi(p, q)$ is a name for an f -solution to p .



We call Φ and Ψ **forward** and **backward** functionals respectively.

Relationships between $C_{\mathbb{N}^{\mathbb{N}}}$ and DS

Our first reduction shows that $DS \leq_W C_{\mathbb{N}^{\mathbb{N}}}$.

Our second reduction is not a Weihrauch reduction from $C_{\mathbb{N}^{\mathbb{N}}}$ to DS because our paths are obtained by applying the limit to descending sequences in $KB(T)$. Nonetheless:

Proposition

$C_{\mathbb{N}^{\mathbb{N}}} \equiv_W \text{lim} * DS$.

$f * g$ is the compositional product (Brattka, Gherardi, Marcone), which captures what can be achieved by first applying g , followed by some computation, and then applying f .

Question (which we will answer)

Do we have $C_{\mathbb{N}^{\mathbb{N}}} \equiv_W DS$? Equivalently, does $C_{\mathbb{N}^{\mathbb{N}}} \leq_W DS$?

Our results: $C_{\mathbb{N}^{\mathbb{N}}} \not\leq_W DS$ (and more)

We show that DS is quite weak in terms of uniform computational strength.

Theorem (G., Pauly, Valenti)

A single-valued problem is Weihrauch reducible to DS if and only if it is Weihrauch reducible to lim , i.e.,

$$\sup_{\leq_W} \{f_0 : \subseteq Z \rightarrow \mathbb{N}^{\mathbb{N}} \mid f_0 \leq_W DS\} \equiv_W \text{lim}.$$

Corollary

The following problems are not Weihrauch reducible to DS:
 LPO' , ADS , lim' , $UC_{\mathbb{N}^{\mathbb{N}}}$, $C_{\mathbb{N}^{\mathbb{N}}}$.

Open question

Is KL (König's lemma) Weihrauch reducible to DS?

Our results: first-order part of DS

Our techniques characterize the problems which have codomain \mathbb{N} and are reducible to DS:

Definition

Let $\Pi_1^1\text{-Bound} : \subseteq \Pi_1^1(\mathbb{N}) \rightrightarrows \mathbb{N}$ be the following problem:
given a Π_1^1 -code for a finite subset of \mathbb{N} , produce a bound.

Theorem (G., Pauly, Valenti)

$\sup_{\leq_W} \{f_0 : \subseteq Z \rightrightarrows \mathbb{N} \mid f_0 \leq_W \text{DS}\} \equiv_W \Pi_1^1\text{-Bound}$.

Dzhafarov, Solomon, Yokoyama (ta) were the first to define and study the **first-order part** 1f of an arbitrary problem f :

$${}^1f \equiv_W \sup_{\leq_W} \{f_0 : \subseteq Z \rightrightarrows \mathbb{N} \mid f_0 \leq_W f\}.$$

Proof that ${}^1\text{DS} \leq_W \Pi_1^1\text{-Bound}$

Suppose $f : \subseteq Z \rightrightarrows \mathbb{N}$ reduces to DS. Given an f -instance p , we can find an f -solution to p as follows.

- 1 At stage s , we can compute a finite piece L_s of the DS-instance defined by the forward functional.
- 2 List all descending sequences in L_s on which the backward functional converges (and hence gives a potential f -solution).
- 3 If such descending sequences exist, we can guess an f -solution by picking the L_s -rightmost descending sequence F_s .
- 4 The set of s such that F_s is undefined or not extendible is Π_1^1, p .
- 5 Apply $\Pi_1^1\text{-Bound}$ to obtain an s such that F_s is extendible. This yields an f -solution to p .

Hence $f \leq_W \Pi_1^1\text{-Bound}$.

Π_1^1 -Bound and Σ_1^1 choice principles

Σ_1^1 - $C_{\mathbb{N}}$: given a Σ_1^1 -code for a nonempty subset of \mathbb{N} , produce an element of the set.

Σ_1^1 - $C_{\mathbb{N}}^{\text{cof}}$: given a Σ_1^1 -code for a cofinite subset of \mathbb{N} , produce an element of the set.

It is easy to see that Σ_1^1 - $C_{\mathbb{N}}^{\text{cof}}$ is Weihrauch equivalent to Π_1^1 -Bound.

Theorem (Angles d'Auriac, Kihara ta)

$\widehat{\Sigma_1^1$ - $C_{\mathbb{N}}^{\text{cof}}} <_{\text{W}} \widehat{\Sigma_1^1$ - $C_{\mathbb{N}}}$, hence Σ_1^1 - $C_{\mathbb{N}}^{\text{cof}} <_{\text{W}} \Sigma_1^1$ - $C_{\mathbb{N}}$.

It is easy to see that the first-order part of $C_{\mathbb{N}^{\mathbb{N}}}$ is Σ_1^1 - $C_{\mathbb{N}}$, so this theorem and our results imply that DS and $C_{\mathbb{N}^{\mathbb{N}}}$ can be separated by considering their first-order parts.

In fact our results imply that there is a single-valued problem with codomain 2 which separates DS and $C_{\mathbb{N}^{\mathbb{N}}}$, namely LPO' .

More about $\widehat{\Sigma_1^1-C_N}$

Kihara, Marcone, Pauly asked if $\widehat{\Sigma_1^1-C_N} <_W C_{\mathbb{N}^{\mathbb{N}}}$.

Theorem (Angles d'Auriac, Kihara ta)

$\widehat{\Sigma_1^1-C_N} <_W C_{\mathbb{N}^{\mathbb{N}}}$. In fact $ATR_2 \not\leq_W \widehat{\Sigma_1^1-C_N}$.

AK proved the above separation using a pair of inseparable Π_1^1 sets.

We will extend their techniques to prove a stronger result about a strengthening of DS.

Definition (G.)

Let $ATR_2 : LO \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be the following problem: given a linear ordering L , produce either an infinite descending sequence in L or a jump hierarchy on L (with a bit indicating which type of solution we produce).

In fact it suffices to consider ATR_2 restricted to computable linear orderings.

ATR₂ and a strengthening of DS

Our earlier results imply that $\text{ATR}_2 \not\leq_W \text{DS}$, but much more is true:

Definition

Let $\Sigma_1^1\text{-DS} : \subseteq \Sigma_1^1(\text{LO}) \Rightarrow \mathbb{N}^{\mathbb{N}}$ be the following problem: given a Σ_1^1 -code for an ill-founded linear ordering, produce an infinite descending sequence.

Theorem (G., Pauly, Valenti)

$\text{ATR}_2 \not\leq_W \Sigma_1^1\text{-DS}$, hence $\Sigma_1^1\text{-DS} <_W C_{\mathbb{N}^{\mathbb{N}}}$.

This means that $C_{\mathbb{N}^{\mathbb{N}}}$ does not reduce to DS even if we allow the forward functional to be Σ_1^1 rather than computable.

On the other hand, we saw earlier that if we allow the backward functional to be limit-computable, then $C_{\mathbb{N}^{\mathbb{N}}}$ reduces to DS.

ATR₂ and inseparable Π_1^1 sets

We can think of ATR₂ as a “union” of DS and

JH: given a linear ordering which supports a jump hierarchy, produce a jump hierarchy.

It is well known that the set of indices of ill-founded linear orderings is Σ_1^1 -complete.

Harrington (unpublished) showed that the set of indices of linear orderings which support a jump hierarchy is also Σ_1^1 -complete.

Theorem (G., generalizing Harrington's proof)

Any Σ_1^1 set which separates wf and hds is Σ_1^1 -complete.

(wf and hds are the set of indices for well-founded linear orderings and linear orderings with hyp descending sequences respectively.)

This generalizes Harrington's result because if a computable linear ordering has a hyp descending sequence, then it does not support a jump hierarchy (Friedman).

Theorem (G., generalizing Harrington's proof)

Any Σ_1^1 set which separates wf and hds is Σ_1^1 -complete.
(wf and hds are the set of indices for well-founded linear orderings and linear orderings with hyp descending sequences respectively.)

By Σ_1^1 -separation,

Corollary

wf and hds cannot be separated by disjoint Σ_1^1 sets.

Angles d'Auriac, Kihara used the corollary to prove that
 $ATR_2 \not\leq_W \widehat{\Sigma_1^1-C_N}$.

We will use the corollary to prove that $ATR_2 \not\leq_W \Sigma_1^1$ -DS.

Proof that $\text{ATR}_2 \not\leq_W \Sigma_1^1\text{-DS}$ (G., Pauly, Valenti)

Suppose that $\text{ATR}_2 \leq_W \Sigma_1^1\text{-DS}$. For each computable linear ordering L_e , the forward functional produces a Σ_1^1 -code $\Phi(L_e)$ for an ill-founded linear ordering.

For the same e , the backward functional may produce either descending sequences in L_e or jump hierarchies on L_e , depending on which descending sequence in $\Sigma_1^1\text{-DS}(\Phi(L_e))$ is given. However, descending sequences are sufficiently homogeneous so

Lemma

For each e , either $\text{DS}(L_e)$ or $\text{JH}(L_e)$ is Muchnik reducible to $\Sigma_1^1\text{-DS}(\Phi(L_e))$.

Then we have disjoint Σ_1^1 sets which separate wf and hds:

$$\text{wf} \subseteq \{e \in \mathbb{N} : \text{DS}(L_e) \text{ is not Muchnik reducible to } \Sigma_1^1\text{-DS}(\Phi(L_e))\}$$

$$\text{hds} \subseteq \{e \in \mathbb{N} : \text{JH}(L_e) \text{ is not Muchnik reducible to } \Sigma_1^1\text{-DS}(\Phi(L_e))\}.$$

This contradicts the corollary on the previous slide. **Thank you!**