Effective embeddings and interpretations

Computable Theory and Applications Seminar

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Classification problem in classes of structures

There is a body of work in mathematical logic dealing with comparing the complexity of the classification problem for various classes of structures.

- (Model Theory) By looking at *the cardinality of the set of isomorphism types*, we know that the classification problem for the class of countable linear orderings ($2^{\aleph_0}$ many isomorphism types) must be more complicated than the classification problem for the class of Q-vector spaces ($\aleph_0$ many isomorphism types)
- (Descriptive Set Theory) Using *Borel embeddings* and the induced partial ordering $\leq_B$. For instance, we know that the class of Abelian p-groups of length ω lies strictly below the class of countable linear orderings in the $\leq_B$ partial ordering.

Borel embedding

Definition (Friedman, Stanley, 1989)

We say that a class $\mathcal{K}$ of structures is *Borel embeddable* in a class of structures $\mathcal{K}'$, and we write $\mathcal{K} \leq_B \mathcal{K}'$, if there is a Borel function $\Phi : K \to K'$ such that for $\mathcal{A}, \mathcal{B} \in K$, $\mathcal{A} \cong \mathcal{B}$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{B})$.

The notion of Borel embedding gives a partial ordering $\leq_B$. If any class of structures could be Borel embedded in a class $\mathcal{K}$ we say that $\mathcal{K}$ is on the top of $\leq_B$.

Note: We could have a uniform Borel procedure for coding structures from structures of class $\mathcal{K}$ in structures from $\mathcal{K}'$. As we shall see, there may or may not be a Borel decoding procedure.

On top under $\leq_B$

Theorem

The following classes lie on top under $\leq_B$.

1. undirected graphs (Lavrov,1963; Nies, 1996; Marker, 2002)
2. fields of any fixed characteristic (Friedman-Stanley; R. Miller-Poonen-Schoutens-Shlapentokh, 2018)
3. 2-step nilpotent groups (Mekler, 1981; Mal'tsev, 1949)
4. linear orderings (Friedman-Stanley)

Computable and Turing computable embeddings

Calvert-Cummins-Knight-S. Miller (2004)
Knight-S. Miller-Vanden Boom (2007)

Definition

We say that a class $\mathcal{K}$ is *Turing computably embedded* in a class $\mathcal{K}'$, and we write $\mathcal{K} \leq_{tc} \mathcal{K}'$, if there is a Turing operator $\Phi : \mathcal{K} \to \mathcal{K}'$ such that for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, $\mathcal{A} \cong \mathcal{B}$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{B})$.

Definition

We say that a class $\mathcal{K}$ is *computably embedded* in a class $\mathcal{K}'$, and we write $\mathcal{K} \leq_{c} \mathcal{K}'$, if there is an enumeration operator $\Psi : \mathcal{K} \to \mathcal{K}'$ such that for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, $\mathcal{A} \cong \mathcal{B}$ iff $\Psi(\mathcal{A}) \cong \Psi(\mathcal{B})$.

The notions of Turing computable embedding and the computable embedding capture in a precise way the idea of uniform effective coding.

The following classes lie on top under $\leq_{tc}$.

1. undirected graphs
2. fields of any fixed characteristic
3. 2-step nilpotent groups
4. linear orderings

The Borel embeddings of Friedman and Stanley, R. Miller, Poonen, Schoutens and Shlapentokh, Lavrov, Nies, Marker, Mekler, and Mal'tsev are all, in fact, Turing computable.

Completeness for degree spectrum and dimensions

Hirschfeldt - Khoussainov - Shore - Slinko, 2002.

A class of structures $\mathcal{K}$ is complete with respect to degree spectra, effective dimensions, expansion by constants, and degree spectra of relations (HKSS-complete) if for every structure $\mathcal{B}$ (in a computable language), there is a structure $\mathcal{A} \in \mathcal{K}$ with the following properties:

1. $DS(\mathcal{A}) = DS(\mathcal{B})$.
2. If $\mathcal{B}$ is computably presentable, then the following holds:
   1. $\mathcal{A}$ has the same **d**-computable dimension as $\mathcal{B}$.
   2. If $b \in \mathcal{B}$, there is an $a \in \mathcal{A}$ such that $(\mathcal{A}, a)$ has the same computable dimension as $(\mathcal{B}, b)$.
   3. If $S \subseteq \mathcal{B}$, there exists $U \subseteq \mathcal{A}$ such that $DS_{\mathcal{A}}(U) = DS_{\mathcal{B}}(S)$ and if S is intrinsically c.e., then so is U.

The *undirected graphs, partial orderings, lattices, rings (with zero-divisors), integral domains of arbitrary characteristic, commutative semigroups, and 2-step nilpotent groups* are all HKSS-complete.

Directed graphs $\leq_{tc}$ undirected graphs

Example (Marker)

For a directed graph G the undirected graph $\Theta(G)$ consists of the following:

1. For each point a in G, $\Theta(G)$ has a point b_a connected to a triangle.
2. For each ordered pair of points $(a; a')$ from G, $\Theta(G)$ has a special point $p_{(a,a')}$ connected directly to b_a and with one stop to b'_a .
3. The point $p_{(a,a')}$ is connected to a square if there is an arrow from a to a', and to a pentagon otherwise.

For structures $\mathcal{A}$ with more relations, the same idea works.

Decoding via nice defining formulas

Fact: For Marker's embedding Θ, we have finitary existential formulas that, for all directed graphs G, define in $\Theta(G)$ the following.

1. the set of points b_a connected to a triangle,
2. the set of ordered pairs such that the special point $p_{(a,a')}$ is part of a square,
3. the set of ordered pairs $(b_a, b_{a'})$ such that the special point $p_{(a,a')}$ is part of a pentagon.

This guarantees a uniform effective procedure that, for any copy of $\Theta(G)$, computes a copy of G. We have uniform effective decoding.

Medvedev reducibility and Decoding

A *problem* is a subset of 2^ω or ω^ω.
A problem P is Medvedev reducible to a problem Q if there is a Turing operator Φ that takes elements of Q to elements of P.

Definition

We say that $\mathcal{A}$ is *Medvedev reducible* to $\mathcal{B}$, and we write $\mathcal{A} \leq_s \mathcal{B}$, if there is a Turing operator that takes copies of $\mathcal{B}$ to copies of $\mathcal{A}$.

Supposing that $\mathcal{A}$ is coded in $\mathcal{B}$, a Medvedev reduction of $\mathcal{A}$ to $\mathcal{B}$ represents an effective decoding procedure.

For classes $\mathcal{K}$ and $\mathcal{K}'$, suppose that $\mathcal{K} \leq_{tc} \mathcal{K}'$ via Θ. A uniform effective decoding procedure is a Turing operator Φ s.t. for all $\mathcal{A} \in \mathcal{K}$, Φ takes copies of $\Theta(\mathcal{A})$ to copies of $\mathcal{A}$.

Effective interpretability

Definition (Montlbán)

A structure $\mathcal{A} = (A, R_i)$ is *effectively interpreted* in a structure $\mathcal{B}$ if there is a set $D \subseteq \mathcal{B}^{<\omega}$ and relations $\sim$ and R_i^* on D, such that

1. $(D, R_i^*)/_\sim \cong \mathcal{A}$,
2. there are computable Σ_1-formulas with no parameters defining a set $D \subseteq \mathcal{B}^{<\omega}$ and relations $(\neg) \sim$ and $(\neg) R_i^*$ in $\mathcal{B}$ (effectively determined).

Example

The usual definition of the ring of integers $\mathbb{Z}$ involves an interpretation in the semi-ring of natural numbers $\mathbb{N}$. Let D be the set of ordered pairs (m, n) of natural numbers. We think of the pair (m, n) as representing the integer $m - n$. We can easily give finitary existential formulas that define ternary relations of addition and multiplication on D, and the complements of these relations, and a congruence relation $\sim$ on D, and the complement of this relation, such that $(D, +, \cdot)/_\sim \cong \mathbb{Z}$.

Computable functor

Definition (R. Miller)

A *computable functor* from $\mathcal{B}$ to $\mathcal{A}$ is a pair of Turing operators Φ, Ψ such that Φ takes copies of $\mathcal{B}$ to copies of $\mathcal{A}$ and Ψ takes isomorphisms between copies of $\mathcal{B}$ to isomorphisms between the corresponding copies of $\mathcal{A}$, so as to preserve identity and composition.

More precisely, Ψ is defined on triples $(\mathcal{B}_1, f, \mathcal{B}_2)$, where $\mathcal{B}_1, \mathcal{B}_2$ are copies of $\mathcal{B}$ with $\mathcal{B}_1 \cong_f \mathcal{B}_2$.

Equivalence

The main result gives the equivalence of the two definitions.

Theorem (Harrison-Trainor - Melnikov - R. Miller - Montalbán 2017)

For structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}$ is effectively interpreted in $\mathcal{B}$ iff there is a computable functor Φ, Ψ from $\mathcal{B}$ to $\mathcal{A}$.

Note: In the proof, it is important that D consist of tuples of arbitrary arity.

Corollary

If $\mathcal{A}$ is effectively interpreted in $\mathcal{B}$, then $\mathcal{A} \leq_s \mathcal{B}$.

Coding and Decoding

Proposition (Kalimullin, 2010)

There exist $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \leq_s \mathcal{B}$ but $\mathcal{A}$ is not effectively interpreted in $\mathcal{B}$.
There exist $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ is effectively interpreted in $(\mathcal{B}, \bar{b})$ but $\mathcal{A}$ is not effectively interpreted in $\mathcal{B}$.

Proposition

If $\mathcal{A}$ is computable, then it is effectively interpreted in all structures $\mathcal{B}$.

Borel interpretability

Harrison-Trainor - R. Miller - Montlbán, 2018, defined Borel versions of the notion of effective interpretation and computable functor.

Definition

1. For a Borel interpretation of $\mathcal{A} = (A, R_i)$ in $\mathcal{B}$ the set $D \subseteq \mathcal{B}^{<\omega}$ the relations $\sim$ and R_i^* on D, are definable by formulas of $L_{\omega_1\omega}$.
2. For a Borel functor from $\mathcal{B}$ to $\mathcal{A}$, the operators Φ and Ψ are Borel.

Note if $R \subseteq \mathcal{B}^{<\omega}$, and we have a countable sequence of $L_{\omega_1\omega}$-formulas $\varphi_n(\bar{x}_n)$ defining $R \cap \mathcal{B}^n$, then we refer to $\bigvee_n \varphi_n(\bar{x}_n)$ as an $L_{\omega_1\omega}$ definition of R.

Their main result gives the equivalence of the two definitions.

Theorem

A structure $\mathcal{A}$ is interpreted in $\mathcal{B}$ using $L_{\omega_1\omega}$-formulas iff there is a Borel functor Φ, Ψ from $\mathcal{B}$ to $\mathcal{A}$.

Graphs and linear orderings

Graphs and linear orderings both lie on top under Turing computable embeddings.

Graphs also lie on top under effective interpretation.

Question: What about linear orderings under effective interpretation?

And under using $L_{\omega_1\omega}$-formulas?

Interpreting graphs in linear orderings

Proposition (Knight-S.-Vatev)

There is a graph G such that for all linear orderings L, $G \not\leq_s L$.

Proof.

Let S be a non-computable set. Let G be a graph such that every copy computes S.

We may take G to be a "daisy" graph", consisting of a center node with a "petal" of length $2n+3$ if $n \in S$ and $2n+4$ if $n \notin S$.

Now, apply:

Proposition (Richter)

For a linear ordering L, the only sets computable in all copies of L are the computable sets.

□

Interpreting a graph in the jump of linear ordering

Proposition (Knight-S.-Vatev)

There is a graph G such that for all linear orderings L, $G \not\leq_s L'$.

Proof.

Let S be a non-Δ_2^0 set. Let G be a graph such that every copy computes S. Then apply:

Proposition (Knight, 1986)

For a linear ordering L, the only sets computable in all copies of L' (or in the jumps of all copies of L), are the Δ_2^0 sets.

□

Interpreting a graph in the second jump of linear ordering

Proposition

For any set S, there is a linear ordering L such that for all copies of L, the second jump computes S.

We may take L to be a "shuffle sum" of the discrete order of type $n+1$ for every $n \in S \oplus S^c = \{2k \mid k \in S\} \cup \{2k+1 \mid k \notin S\}$ and order type ω (densely many copies of each of these orderings). Then we have a pair of finitary Σ_3 formulas saying that $n \in S$ if L has a maximal discrete set of size $2n+1$ and $n \notin S$ if L has a maximal discrete set of size $2n+2$. It follows that any copy of L'' uniformly computes the set S.

Proposition (Knight-S.-Vatev)

For any graph G, there is a linear ordering L such that $G \leq_s L''$.

Let S be the diagram of a specific copy of G and let L be a linear order such that $S \leq_s L''$. Then $G \leq_s L''$.

Friedman-Stanley embedding of graphs in orderings

Friedman and Stanley determined a Turing computable embedding $L : G \to L(G)$, where $L(G)$ is a sub-ordering of $Q^{<\omega}$ under the lexicographic ordering.

1. Let $(A_n)_{n\in\omega}$ be an effective partition of $\mathbb{Q}$ into disjoint dense sets.
2. Let $(t_n)_{1\leq n}$ be a list of the atomic types in the language of directed graphs.

Definition

For a graph G, the elements of $L(G)$ are the finite sequences $r_0 q_1 r_1 \ldots r_{n-1} q_n r_n k \in \mathbb{Q}^{<\omega}$ such that for $i < n$, $r_i \in A_0$, $r_n \in A_1$, and for some $a_1, \ldots, a_n \in G$, satisfying t_m, $q_i \in A_{a_i}$ and $k < m$.

No uniform interpretation of G in $L(G)$

Theorem (Knight-S.-Vatev)

There are no $L_{\omega_1\omega}$ formulas that, for all graphs G, interpret G in $L(G)$.

The idea of Proof: We may think of an ordering as a directed graph. It is enough to show the following.

Proposition

A ω_1^{CK} is not interpreted in $L(\omega_1^{CK})$ using computable infinitary formulas.

B For all X, ω_1^X is not interpreted in $L(\omega_1^X)$ using X-computable infinitary formulas.

Proof of A

The Harrison ordering H has order type $\omega_1^{CK}(1+\eta)$. It has a computable copy.

Let I be the initial segment of H of order type ω_1^{CK}. Thinking of H as a directed graph, we can form the linear ordering $L(H)$. We consider $L(I) \subseteq L(H)$.

Lemma

$L(I)$ is a computable infinitary elementary substructure of $L(H)$.

Proposition (Main)

There do not exist computable infinitary formulas that define an interpretation of H in $L(H)$ and an interpretation of I in $L(I)$.

To prove A, we suppose that there are computable infinitary formulas interpreting ω_1^{CK} in $L(\omega_1^{CK})$. Using Barwise Compactness theorem, we get essentially H and I with these formulas interpreting H in $L(H)$ and I in $L(I)$.

Proof of the Proposition(Main)

Lemma

1. For any $\bar{b} \in L(I)$, and $c \in L(I)$ there is an automorphism of $L(I)$ taking $\bar{b}$ to a tuple $\bar{b}'$ entirely to the right of c.
2. For any $\bar{b} \in L(I)$, and $c \in L(I)$ there is also an automorphism taking $\bar{b}$ to a tuple $\bar{b}''$ entirely to the left of c.

Lemma

Suppose that we have computable Σ_γ formulas D, $\otimes$ and $\sim$, defining an interpretation of H in $L(H)$ and I in $L(I)$. Then in $D^{L(I)}$ there is a fixed n, and there are n-tuples, all satisfying the same Σ_γ formulas, and representing arbitrarily large ordinals $\alpha < \omega_1^{CK}$.

We arrive at a contradiction by producing tuples $\bar{b}, \bar{b}', \bar{c}$ in $D^{L(I)}$, $\bar{b}$ and $\bar{b}'$ are automorphic, $\bar{b}, \bar{c}$ and $\bar{c}, \bar{b}'$ satisfy the same computable Σ_γ formulas, and the ordinal represented by $\bar{b}$ and $\bar{b}'$ is smaller than that represented by $\bar{c}$. Then $\bar{b}, \bar{c}$ should satisfy $\otimes$, while $\bar{c}, \bar{b}'$ should not.

Conjecture

We believe that Friedman and Stanley did the best that could be done.

Conjecture. For any Turing computable embedding Θ of graphs in orderings, there do not exist $L_{\omega_1 \omega}$ formulas that, for all graphs G, define an interpretation of G in $\Theta(G)$.

M. Harrison-Trainor and A. Montlbán came to a similar result recently by a totally different construction. Their result is that there exist structures which cannot be computably recovered from their tree of tuples. They proved :

1. There is a structure $\mathcal{A}$ with no computable copy such that $T(\mathcal{A})$ has a computable copy.
2. For each computable ordinal α there is a structure $\mathcal{A}$ such that the Friedman and Stanley Borel interpretation $L(\mathcal{A})$ is computable but $\mathcal{A}$ has no Δ^0_α copy.

Mal'tsev embedding of fields in groups

If F is a field, we denote by $H(F)$ the multiplicative group of matrices of kind

$$h(a,b,c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where $a, b, c \in F$. Note that $h(0,0,0) = 1$.
Groups of kind $H(F)$ are known as *Heisenberg groups*.

Theorem (Mal'tsev)

There is a copy of F defined in $H(F)$ with parameters.

Definition of F in $H(F)$

Let u, v be a non-commuting pair in $H(F)$.
Then $(D, +, \cdot_{(u,v)})$ is a copy of F, where

1. D is the group center – $x \in D \iff [x,u] = 1$ and $[x,v] = 1$,
2. $x + y = z$ if $x * y = z$, where $*$ is the group operation,
3. $x \cdot_{(u,v)} y = z$ if there exist x', y' such that
   $[x', u] = [y', v] = 1, [x', v] = x, [u, y'] = y$, and $[x', y'] = z$.

Here $[x, y] = x^{-1}y^{-1}xy$.

Definability: We have finitary existential formulas that define D and the relation $+$ and its complement. For any non-commuting pair (u, v), we have finitary existential formulas, with parameters (u, v) that define the relation $\cdot$ and its complement.

Natural isomorphisms

Let $F_{(u,v)}$ be the copy of F defined in $H(F)$ with a non-commuting pair (u, v), where $u = h(u_1, u_2, u_3)$ and $v = h(v_1, v_2, v_3)$. Let

$$\Delta_{(u,v)} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Theorem

The function f that takes $x \in F$ to $h(0, 0, \Delta_{(u,v)} \cdot_F x)$ is an isomorphism between F and $F_{(u,v)}$.

Computable functor

Proposition

There is a uniform Turing operator that, for all F, takes copies of $H(F)$ to copies of F.

We look for a non-commuting pair (u, v) in $H(F)$, and, for the first we find, take the copy of F defined using these parameters.

Lemma

There is a finitary existential formula $\psi(u, v, u', v', x, y)$ that, for any two non-commuting pairs (u, v) and (u', v'), defines the isomorphism $f_{(u,v),(u',v')}$ taking $x \in F_{(u,v)}$ to $y \in F_{(u',v')}$. Moreover, the family of isomorphisms $f_{(u,v),(u',v')}$ is functorial; i.e.,

1. for any non-commuting pair (u, v), the function $f_{(u,v),(u,v)}$ is the identity,
2. for any three non-commuting pairs (u, v), (u', v'), and (u'', v''),

$$f_{(u,v),(u'',v'')} = f_{(u',v'),(u'',v'')} \circ f_{(u,v),(u',v')}.$$

Defining the interpretation directly

Proposition
(Alvir,Calvert,Goodman,Harizanov,Knight,Miller,Morozov,S,Weisshaar)

There are finitary existential formulas that define a uniform effective interpretation of F in $H(F)$, where the set of tuples from $H(F)$ that represent elements of F has arity 3.

We define $D \subseteq H(F)^3$, binary relations $\pm \sim$ on D, and ternary relations $\oplus$, $\otimes$ (which are binary operations on D), as follows:

1. D is the set of triples (u, v, x) such that (u, v) is a non-commuting pair and x commutes with both u and v.
2. $(u, v, x) \sim (u', v', x')$ holds if the natural isomorphism $f_{(u,v),(u',v')}$ from $F_{(u,v)}$ to $F_{(u',v')}$ takes x to x'.
3. $\oplus((u, v, x), (u', v', y), (u'', v'', z))$ holds if there exist y', z' such that $(u, v, y') \sim (u', v', y)$, $(u, v, z') \sim (u'', v'', z)$, and $F_{(u,v)} \models x + y' = z'$.
4. $\otimes((u, v, x), (u', v', y), (u'', v'', z))$ holds if there exist y', z' such that $(u, v, y') \sim (u', v', y)$, $(u, v, z') \sim (u'', v'', z)$, and $F_{(u,v)} \models xy' = z'$.

A question of bi-interpretability

If $\mathcal{B}$ is interpreted in $\mathcal{A}$, we write $\mathcal{B}^{\mathcal{A}}$ for the copy of $\mathcal{B}$ given by the interpretation of $\mathcal{B}$ in $\mathcal{A}$.

Definition (Effective bi-interpretability)

Structures $\mathcal{A}$ and $\mathcal{B}$ are *effectively bi-interpretable* if we have interpretations of $\mathcal{A}$ in $\mathcal{B}$ and of $\mathcal{B}$ in $\mathcal{A}$ such that there are uniformly relatively intrinsically computable isomorphisms from $\mathcal{A}$ to $\mathcal{A}^{\mathcal{B}^{\mathcal{A}}}$ and from $\mathcal{B}$ to $\mathcal{B}^{\mathcal{A}^{\mathcal{B}}}$.

Question (Montalbán)

Do we have uniform effective bi-interpretability of F and $H(F)$?

The answer to this question is negative. In particular, $\mathbb{Q}$ and $H(\mathbb{Q})$ are not effectively bi-interpretable. One way to see this is to note that $\mathbb{Q}$ is rigid, while $H(\mathbb{Q})$ is not—in particular, for a non-commuting pair, u, v, there is a group automorphism that takes (u, v) to (v, u). In his book Montalbán shows that if $\mathcal{A}$ and $\mathcal{B}$ are effectively bi-interpretable and $\mathcal{A}$ is rigid, then so is $\mathcal{B}$.

Generalizing

Proposition

Suppose $\mathcal{A}$ has a copy $\mathcal{A}_{\bar{b}}$ defined in $(\mathcal{B}, \bar{b})$, using computable Σ_1 formulas, where the orbit of $\bar{b}$ is defined by a computable Σ_1 formula $\varphi(\bar{x})$. Suppose also that there is a computable Σ_1 formula $\psi(\bar{b}, \bar{b}', u, v)$ that, for any tuples $\bar{b}$, $\bar{b}'$ satisfying $\varphi(\bar{x})$, defines a specific isomorphism $f_{\bar{b},\bar{b}'}$ from $\mathcal{A}_{\bar{b}}$ onto $\mathcal{A}_{\bar{b}'}$. We suppose that for each $\bar{b}$ satisfying φ, $f_{\bar{b},\bar{b}}$ is the identity isomorphism, and for any $\bar{b}$, $\bar{b}'$, and $\bar{b}''$ satisfying φ, $f_{\bar{b}',\bar{b}''} \circ f_{\bar{b},\bar{b}'} = f_{\bar{b},\bar{b}''}$. Then there is an effective interpretation of $\mathcal{A}$ in $\mathcal{B}$.

$SL_2(C)$

Let C be an algebraically closed field of characteristic 0 and of infinite transcendence degree.
We consider $SL_2(C)$ for the group of 2×2 matrices over C with determinant 1.

Proposition

C is interpreted in $SL_2(C)$ with parameters.

Let A be the set of matrices of form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$.

Let M be the set of matrices of form $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$.

$SL_2(C)$

Let T consist of the pairs (X, Y) such that $X \in A$ and $Y \in M$ and Y has a square root Z such that $Z * P * Z^{-1} = X$.

For $(X, Y) \in T$, we define addition and multiplication relations as follows:

1. $(X, Y) \oplus (X', Y') = (U, V)$ if $X * X' = U$ and $(U, V) \in T$,
2. $(X, Y) \otimes (X', Y') = (U, V)$ if $Y * Y' = V$ and $(U, V) \in T$.

We define the set T with parameters.

There is an old result, of Poizat, according to Pillay, saying that C is interpreted in $SL_2(C)$ by elementary first order formulas with no parameters. But we do not know the complexity of the defining formulas. We have a formula $\varphi(u, v)$, that give our interpretation of C in $SL_2(C)$ that gives an infinite field $F(u, v)$, not of characteristic 2, in which every element has a square root. The theory of $SL_2(C)$ is ω-stable. By an old result of Macintyre, $F(u, v)$ must be algebraically closed. Poizat's results show that $F(u, v)$ is isomorphic to C and that there are unique definable isomorphisms between the fields $F(u, v)$. So, we have, not necessarily an effective interpretation without parameters, but one that is defied by elementary first order formulas.

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THANK YOU