Automorphism argument and reverse mathematics

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Automorphisms of models of first-order arithmetic

Theorem (Smoryński)

Let $M \models \mathrm{I}\Delta_0$ be countable recursively saturated, let $I \subseteq_e M$ be closed under exp, and let $d > I$.
If $\bar{a}, \bar{b} < d$ satisfy $\mathrm{Th}_{\Delta_0}(\bar{a}, x, d) = \mathrm{Th}_{\Delta_0}(\bar{b}, x, d)$ for any $x \in I$, then there exists an automorphism $h : d \to d$ with fixing I pointwise and $h(\bar{a}) = \bar{b}$.

Given $M \models \mathrm{I}\Delta_0$ and $A \subseteq M$, we write $(M; A) \models \mathrm{I}\Sigma^0_n$ or $\mathrm{I}\Sigma_n(A)$ (resp. $\mathrm{B}\Sigma^0_n$ or $\mathrm{B}\Sigma_n(A)$) if M satisfies Σ_1-induction (resp. bounding) with A as a unary predicate.

Theorem (Essentially Kossak)

If $(M; A) \models \mathrm{B}\Sigma^0_1 + \exp + \neg\mathrm{I}\Sigma^0_1$, then M has continuum many automorphisms.

Weak fragments of second-order arithmetic

In this talk, we care about the hierarchy of induction and bounding.

- EFA: "discrete ordered semi-ring"+ "2^x is total"+ Σ_0-induction.
- $\mathrm{I}\Sigma^0_n$: EFA+Σ^0_n induction.
- $\mathrm{B}\Sigma^0_n$: EFA+Σ^0_n bounding:

$$\forall u\,(\forall x < u\,\exists y\,\psi(x,y) \to \exists v\,\forall x < u\,\exists y < v\,\psi(x,y))$$

Kirby-Paris hierarchy: $\mathrm{B}\Sigma^0_1 < \mathrm{I}\Sigma^0_1 < \mathrm{B}\Sigma^0_2 < \mathrm{I}\Sigma^0_2 < \mathrm{B}\Sigma^0_3 < \dots$

And, I would like to talk about models of...

- RCA^*_0: EFA+Σ^0_0 induction+Δ^0_1 comprehension.
- RCA_0: RCA^*_0+Σ^0_1 induction.
- WKL^*_0: RCA^*_0 + weak König's lemma.
- WKL_0: RCA_0 + weak König's lemma.
- $\mathsf{RCA}_0 + \mathrm{COH} + \mathrm{B}\Sigma^0_2$, $\mathsf{RCA}_0 + \mathrm{RT}^2_2$,...

Coded submodels

An $\mathcal{L}_2$-structure is a pair $(M, \mathcal{X})$, where M is an $\mathcal{L}_1$-structure and $\mathcal{X} \subseteq \mathcal{P}(M)$.

- In this talk, we mostly consider models $(M, \mathcal{X}) \models \mathrm{RCA}_0^*$, and always assume that both of M and $\mathcal{X}$ are countable.

Definition

Let $(M, \mathcal{X}) \models \mathrm{RCA}_0^*$. A coded submodel (or c.c.ω-submodel) is a set $\mathcal{W} = \{W_k : k \in M\} \in \mathcal{X}$, where $x \in W_k \leftrightarrow (x, k) \in \mathcal{W}$. Then, $(M, \mathcal{W})$ is an (ω-)substructure of $(M, \mathcal{X})$.

- Note that a coded submodel is not equipped with its truth definition.
- However, $(M, \mathcal{W}) \models \psi$ can be always described by an arithmetical formula within $(M, \mathcal{X})$.

Coded submodels

We will use the following theorem throughout this talk.

Theorem

Let $(M, \mathcal{X}) \models \mathrm{RCA}_0^*$. Then the following are equivalent.

1. $(M, \mathcal{X}) \models \mathrm{WKL}_0^*$.
2. *For any $A \in \mathcal{X}$, there exists a coded submodel $\mathcal{W} \in \mathcal{X}$ such that $A \in \mathcal{W}$ and $(M, \mathcal{W}) \models \mathrm{WKL}_0^*$.*

- We often consider models $(M, \mathcal{X}) \models \mathrm{WKL}_0^* + \mathrm{B}\Sigma_n^0 + \neg\mathrm{I}\Sigma_n^0$.
- Since $\neg\mathrm{I}\Sigma_n^0$ is a Σ_1^1-statement, we may fix $A_0 \in \mathcal{X}$ such that $(M; A_0) \models \neg\mathrm{I}\Sigma_n(A_0)$.
- Then, any coded submodel which contains A_0 satisfies $\mathrm{B}\Sigma_n^0 + \neg\mathrm{I}\Sigma_n^0$.
- Moreover, if $n = 1$, $A_0 \subseteq M$ can be taken as an unbounded set which is described by an increasing enumeration $A_0 = \{d_i : i \in I\}$ where $I = \mathrm{card}_M(A_0)$ is a proper Σ_1^0-definable cut.

Automorphism argument for WKL_0^*

Theorem

Let $(M, \mathcal{X})$ be a countable recursively saturated model of $\mathrm{WKL}_0^* + \neg \mathrm{I}\Sigma_1^0$, and let $\mathcal{W} = \{W_k : k \in M\} \in \mathcal{X}$ be a coded submodel of $(M, \mathcal{X})$ satisfying $\mathrm{WKL}_0^* + \neg \mathrm{I}\Sigma_1^0$.
Then, for any $\bar{a} \in M$ and $\bar{A} \in \mathcal{W}$, there exists an isomorphism h between $(M, \mathcal{X})$ and $(M, \mathcal{W})$ such that $h(\bar{a}) = \bar{a}$ and $h(\bar{A}) = \bar{A}$.

- $((M, \mathcal{X}), (M, \mathcal{W}))$ form a recursively saturated pair.
- $h : M \to M$ is just an automorphism, and will not be identity.

Idea of the proof.
Build the isomorphism h by a B & F as follows:
take $A_0 = \{d_i : i \in I\} \in \mathcal{W}$ where $I = \mathrm{card}_M(A_0)$ is a proper cut and $I < b$, and then construct h so that
for each $\bar{r}, \bar{s} \in M$, $\bar{R} \in \mathcal{X}$, $\bar{S} \in \mathcal{W}$, $h(\bar{r}) = \bar{s}$, $h(\bar{R}) = h(\bar{S})$ implies

(#) for each Δ_0 formula δ, $n \in \omega$, $j \leq \exp_n(b)$, and $i \in I$,
$(M, \mathcal{X}) \models \delta(d_i, j, \bar{r}, \bar{R}) \leftrightarrow \delta(d_i, j, \bar{s}, \bar{S})$.

Automorphism argument for WKL_0^*

Theorem

Let $(M, \mathcal{X})$ be a countable recursively saturated model of $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma_1^0$, and let $\mathcal{W} = \{W_k : k \in M\} \in \mathcal{X}$ be a coded submodel of $(M, \mathcal{X})$ satisfying $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma_1^0$.
Then, for any $\bar{a} \in M$ and $\bar{A} \in \mathcal{W}$, there exists an isomorphism h between $(M, \mathcal{X})$ and $(M, \mathcal{W})$ such that $h(\bar{a}) = \bar{a}$ and $h(\bar{A}) = \bar{A}$.

Corollary

Let $(M, \mathcal{X})$ be a model of $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma_1^0$, and let $\mathcal{W} \in \mathcal{X}$ be code a submodel of $(M, \mathcal{X})$ satisfying $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma_1^0$.
Then, $(M, \mathcal{W})$ is an elementary submodel of $(M, \mathcal{X})$.

We will see some (weird) consequences of this theorem.

Collapsing analytic hierarchy

Within $\mathrm{WKL}_0^* + \neg \mathrm{I}\Sigma_1^0$, analytic hierarchy collapses.

Proposition

Let $(M, \mathcal{X}) \models \mathrm{WKL}_0^* + \neg \mathrm{I}\Sigma_1^0$, and let $\bar{a} \in M$, $\bar{A} \in \mathcal{X}$. For any $\mathcal{L}_2$-formula $\psi(\bar{x}, \bar{X})$, the following are equivalent.

- $(M, \mathcal{X}) \models \psi(\bar{a}, \bar{A})$.
- *For any coded submodel $\mathcal{W} = \{W_k : k \in M\} \in \mathcal{X}$ with $\bar{A} \in W$, $(M, \mathcal{W}) \models \psi(\bar{a}, \bar{A})$.*
- *There exists a coded submodel $\mathcal{W} = \{W_k : k \in M\} \in \mathcal{X}$ with $\bar{A} \in W$ such that $(M, \mathcal{W}) \models \psi(\bar{a}, \bar{A})$.*

Corollary

Within $\mathrm{WKL}_0^* + \neg \mathrm{I}\Sigma_1^0$, any $\mathcal{L}_2$-formula $\psi(\bar{x}, \bar{X})$ is equivalent to a Π_1^1-formula and a Σ_1^1-formula.

Undefinability of coded models

Over $\mathsf{RCA}_0^* + \neg \mathrm{I}\Sigma_1^0$, low basis theorem fails badly...

- Let $(M, \mathcal{X}) \models \mathsf{WKL}_0^* + \neg \mathrm{I}\Sigma_1^0$ and let $\mathcal{W} \in \mathcal{X}$ be a coded submodel satisfying $\mathsf{WKL}_0^* + \mathrm{I}\Sigma_1^0$.
- Assume that $\mathcal{W} (\subseteq M)$ is definable in $(M, \mathcal{W})$ as $\mathcal{W} = \{a \in M : (M, \mathcal{W}) \models \theta(a)\}$.
- Then, $(M, \mathcal{X}) \models \exists Z (Z = \theta(\mathbb{N}))$ but $(M, \mathcal{W}) \models \neg \exists Z (Z = \theta(\mathbb{N}))$, which is a contradiction.

Corollary

If $(M, \mathcal{X}) \models \mathsf{WKL}_0^*$ and $\mathcal{X} \subseteq \mathrm{ARITH}(M)$, then $(M, \mathcal{X}) \models \mathsf{WKL}_0$.

Corollary

The following are equivalent over RCA_0^*.

1. $\mathrm{I}\Sigma_1^0$.
2. *For any infinite tree $T \subseteq 2^{<\mathbb{N}}$, there exists a $\Sigma_n(T)$-definable path of T which satisfies $\mathrm{B}\Sigma_1^0$ ($n \geq 2$).*

Maximal Π^1_1-conservative extension

Theorem

- *(Simpson/Smith)* WKL_0^* *is* Π^1_1*-conservative over* RCA_0^*.
- *Let* $T \supseteq \mathrm{RCA}_0^*$ *be a* Π^1_2*-theory. If* φ, ψ *are* Π^1_2*-sentences and* $T + \varphi$ *and* $T + \psi$ *are both* Π^1_1*-conservative over* T*, then* $T + \varphi + \psi$ *is* Π^1_1*-conservative over* T*.*

- Assume that a Π^1_2-sentence φ is Π^1_1-conservative over $\mathrm{RCA}_0^* + \neg\mathrm{I}\Sigma^0_1$.
- Then $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma^0_1 + \varphi$ is also Π^1_1-conservative over $\mathrm{RCA}_0^* + \neg\mathrm{I}\Sigma^0_1$.
- Within $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma^0_1$, φ is equivalent to a Π^1_1-sentence $\tilde{\varphi}$.
- Then $\mathrm{RCA}_0^* + \neg\mathrm{I}\Sigma^0_1 \vdash \tilde{\varphi}$, and thus $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma^0_1 \vdash \varphi$.

Corollary

A Π^1_2*-sentence* ψ *is* Π^1_1*-conservative over* $\mathrm{RCA}_0^* + \neg\mathrm{I}\Sigma^0_1$ *if and only if* $\mathrm{WKL}_0^* + \neg\mathrm{I}\Sigma^0_1$ *proves* ψ*.*

Generalization to $\mathrm{RCA}_0^* + \mathrm{B}\Sigma_n^0 + \neg\mathrm{I}\Sigma_n^0$

Let $(M, \mathcal{X}) \models \mathrm{RCA}_0^* + \mathrm{B}\Sigma_n^0$.

- Given $A, B \in \mathcal{X}$, we write $A \ll_{\Delta_n} B$ if there exists a $\Delta_n(M; B)$-definable coded submodel of WKL_0^* containing $A^{(n-1)}$, or equivalently all $\Delta_n(M; A)$ sets.

Definition

$\Delta_n\mathrm{WKL}$: $\forall X \exists Y (X \ll_{\Delta_n} Y)$.

- $(M, \mathcal{X}) \models \mathrm{B}\Sigma_n^0 + \Delta_n\mathrm{WKL} \iff (M, \Delta_n(M, \mathcal{X})) \models \mathrm{WKL}_0^*$.
- $\Delta_2\mathrm{WKL}$ is equivalent to COH over $\mathrm{RCA}_0 + \mathrm{B}\Sigma_2^0$ (Belanger).
- $\mathrm{RCA}_0^* + \mathrm{B}\Sigma_n^0 + \Delta_n\mathrm{WKL}$ is Π_1^1-conservative over $\mathrm{RCA}_0^* + \mathrm{B}\Sigma_n^0$ (essentially Belanger).

Theorem

Let $(M, \mathcal{X}) \models \mathrm{RCA}_0^* + \mathrm{B}\Sigma_n^0 + \neg\mathrm{I}\Sigma_n^0$ be recursively saturated. Let $A, B \in \mathcal{X}$ such that $(M; A) \models \neg\mathrm{I}\Sigma_n(A)$ and $A \ll_{\Delta_n} B$. Then, there exists $\mathcal{Y} \subseteq \Delta_n(M; B)$ such that $A \in \mathcal{Y}$, $(M, \mathcal{X})$ is isomorphic to $(M, \mathcal{Y})$ with fixing A and $\Delta_n(M, \mathcal{Y}) \subseteq \Delta_n(M; B)$.

Π^1_1-part of $\mathsf{RCA}_0 + \mathrm{RT}^2_2$

Theorem (Cholak/Jockusch/Slaman)

$\mathsf{RCA}_0 + \mathrm{RT}^2_2 + \mathrm{I}\Sigma^0_2$ *a* Π^1_1*-conservative extension of* $\mathsf{RCA}_0 + \mathrm{I}\Sigma^0_2$.

- Indeed, any countable model of $(M, \mathcal{X}) \models \mathsf{RCA}_0 + \mathrm{I}\Sigma^0_2$ has an ω-extension $\mathcal{Y} \supseteq \mathcal{X}$ such that $(M, \mathcal{Y}) \models \mathsf{RCA}_0 + \mathrm{RT}^2_2 + \mathrm{I}\Sigma^0_2$.

Question (Cholak/Jockusch/Slaman)

Is $\mathsf{RCA}_0 + \mathrm{RT}^2_2$ a Π^1_1-conservative extension of $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2$?

- To answer to this question, we need to know when $(M, \mathcal{X}) \models \mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2 + \neg\mathrm{I}\Sigma^0_2$ can be extended to a model of $(M, \mathcal{Y}) \models \mathsf{RCA}_0 + \mathrm{RT}^2_2(+\neg\mathrm{I}\Sigma^0_2)$.

Models of $\mathrm{RCA}_0 + \mathrm{RT}^2_2 + \neg\mathrm{I}\Sigma^0_2$

- Let $(M, \mathcal{X}) \models \mathrm{RCA}_0 + \mathrm{RT}^2_2 + \neg\mathrm{I}\Sigma^0_2$ be recursively saturated, and let $A \in \mathcal{X}$ such that $(M; A) \models \neg\mathrm{I}\Sigma_2(A)$.
- Since $\mathrm{RCA}_0 + \mathrm{RT}^2_2 \vdash \mathrm{COH}$, we have $(M, \mathcal{X}) \models \Delta_2\mathrm{WKL}$.
- Let $f \in \mathcal{X}$ be a coloring $f : [M]^2 \to 2$, and let $B \in \mathcal{X}$ such that $f \oplus A \ll_{\Delta_2} B$.
- Then, there exists $\mathcal{Y} \subseteq \Delta_2(M; B)$ such that $f \oplus A \in \mathcal{Y}$, $(M, \mathcal{X})$ is isomorphic to $(M, \mathcal{Y})$ with fixing $f \oplus A$ and $\Delta_2(M, \mathcal{Y}) \subseteq \Delta_2(M; B)$.
- Since $(M, \mathcal{Y}) \models \mathrm{RT}^2_2$, there exists $H \in \mathcal{Y}$ such that H is an infinite homogeneous set for f.
- H is “low” in the sense that $\Delta_2(M; f \oplus X \oplus H) \subseteq \Delta_2(M; B)$.

Theorem

$\mathrm{RCA}_0 + \mathrm{RT}^2_2$ *proves the following:*

(†) $\forall X \forall Y \forall f : [\mathbb{N}]^2 \to 2$
$[\neg\mathrm{I}\Sigma_2(X) \wedge f \oplus X \ll_{\Delta_2} Y \to \exists H \in \Delta_2(\mathbb{N}; Y)($“*H is an infinite homogeneous set for f*”$\wedge \Delta_2(\mathbb{N}; f \oplus X \oplus H) \subseteq \Delta_2(\mathbb{N}; Y))]$.

Π^1_1-part of $\mathrm{RCA}_0 + \mathrm{RT}^2_2$

Theorem

$\mathrm{RCA}_0 + \mathrm{RT}^2_2$ *proves the following:*

($\dagger$) $\forall X \forall Y \forall f : [\mathbb{N}]^2 \to 2$
$[\neg \mathrm{I}\Sigma_2(X) \wedge f \oplus X \ll_{\Delta_2} Y \to \exists H \in \Delta_2(\mathbb{N}; Y)($*"H is an infinite homogeneous set for f"* $\wedge \Delta_2(\mathbb{N}; f \oplus X \oplus H) \subseteq \Delta_2(\mathbb{N}; Y))]$.

- ($\dagger$) is a $\forall\Pi^0_5$-statement.
- ($\dagger$) is the basis theorem for RT^2_2 by the "first-jump control" argument in the Cholak/Jockusch/Slaman paper.
- If $\mathrm{B}\Sigma^0_2$ proves ($\dagger$) then any countable model $(M; A) \models \mathrm{B}\Sigma^0_2$ can be extended to a model of $\mathrm{RCA}_0 + \mathrm{RT}^2_2$.

Corollary

- $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ *is* Π^1_1*-conservative over* $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$ *if and only if* $\mathrm{B}\Sigma^0_2$ *proves* ($\dagger$).
- *If* $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ *is* $\forall\Pi^0_5$*-conservative over* $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$ *then it is* Π^1_1*-conservative over* $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$.

Other consequences

"Undefinability" is also available for the case $n \geq 2$.

- If $(M, \mathcal{X}) \models \mathsf{RCA}_0 + \mathrm{B}\Sigma^0_n + \Delta_n\mathrm{WKL}$ and $\mathcal{X} \subseteq \mathrm{ARITH}(M)$, then $(M, \mathcal{X}) \models \mathsf{RCA}_0 + \mathrm{I}\Sigma^0_n$.
  - Particularly, there is no $(M, \mathcal{X}) \models \mathsf{RCA}_0 + \mathrm{RT}^2_2 + \neg\mathrm{I}\Sigma^0_2$ with $\mathcal{X} \subseteq \mathrm{ARITH}(M)$.

The maximal Π^1_1-conservative extension of $\mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ is c.e.

- Let $\psi \equiv \forall X \exists Y \theta(X, Y)$ be a Π^1_2-sentence. Then, $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n + \psi$ is Π^1_1-conservative over $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ if and only if $\mathrm{B}\Sigma^0_n + \neg\mathrm{I}\Sigma^0_n$ proves

$$(\ddagger)\quad \forall X \forall Y\, [\neg\mathrm{I}\Sigma_n(X) \land X \ll_{\Delta_n} Y \to \exists Z \in \Delta_n(\mathbb{N}; Y)(\theta(X, Z) \land \Delta_n(\mathbb{N}; X \oplus Z) \subseteq \Delta_n(\mathbb{N}; Y))].$$

- WKL^*_0 is conservative over RCA^*_0 w.r.t. formulas of the form $\forall X \exists! Y \theta(X, Y)$ where $\theta \in \Sigma^1_0$.
- (and maybe more)...

Thank you!

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