Generic realizability for intuitionistic set theory

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Outline

- Realizability
- Generic realizability for set theory
- Recent work (joint with Michael Rathjen and Takako Nemoto)

Realizability

Ideology: Brouwer-Heyting-Kolmogorov interpretation.

BHK says that φ is true if there is a proof of φ.

- a proves $\varphi \wedge \psi$ iff a is a pair (a_0, a_1), a_0 proves φ and a_1 proves ψ
- a proves $\varphi \vee \psi$ iff a is a pair (a_0, a_1), and either $a_0 = 0$ and a_1 proves φ or else $a_0 = 1$ and a_1 proves ψ
- a proves $\varphi \to \psi$ iff if b proves φ then $a(b)$ proves ψ for every b
- a proves $\forall x\, \varphi(x)$ iff $a(x)$ proves $\varphi(x)$ for every $x \in D$
- a proves $\exists x\, \varphi(x)$ iff a is a pair (x, b), where $x \in D$ and b proves $\varphi(x)$

Realizability

General framework:

- A is a partial combinatory algebra (pca) with partial application $(a, b) \mapsto a \cdot b$ from $A \times A$ to A
- BHK-like relation $a \Vdash \varphi$ with $a \in A$ and φ formula

Formalized realizability:

- $T + S \vdash \varphi$ implies $T \vdash \exists a \in A\, (a \Vdash \varphi)$

Applications of formalized realizability:

- consistency, independence and conservation results
- metamathematical properties (closure under rules, existence property, etc)

Example: Kleene recursive realizability

Interpretation of HA intuitionistic first order arithmetic.

Kleene first algebra:

- $A = \omega$
- $(a, b) \mapsto \{a\}(b)$

Definition (Kleene recursive realizability)

- $a \Vdash \varphi$ iff φ, for φ atomic.
- $a \Vdash \varphi \wedge \psi$ iff $a = (a_0, a_1)$, $a_0 \Vdash \varphi$ and $a_1 \Vdash \psi$
- $a \Vdash \varphi \vee \psi$ iff $a = (a_0, a_1)$, and either $a_0 = 0$ and $a_1 \Vdash \varphi$ or $a_0 = 1$ and $a_1 \Vdash \psi$
- $a \Vdash \neg\varphi$ iff $b \nVdash \varphi$ for every b
- $a \Vdash \varphi \rightarrow \psi$ iff $b \Vdash \varphi$ implies $\{a\}(b) \Vdash \psi$ for every b
- $a \Vdash \forall n\, \varphi(n)$ iff $\{a\}(n) \Vdash \varphi(n)$ for every n
- $a \Vdash \exists n\, \varphi(n)$ iff $a = (a_0, a_1)$ and $a_1 \Vdash \varphi(a_0)$

Example: Kleene recursive realizability

Definition (Kleene recursive realizability with truth)

- $a \Vdash_{tr} \neg\varphi$ iff $\neg\varphi$
- $a \Vdash_{tr} \varphi \to \psi$ iff $\varphi \to \psi$ and $b \Vdash_{tr} \varphi$ implies $\{a\}(b) \Vdash_{tr} \psi$ for every b
- other clauses as in Kleene recursive realizability

Theorem

$\mathrm{HA} \vdash \exists a\,(a \Vdash_{tr} \varphi) \to \varphi.$

Theorem (soundness)

If $\mathrm{HA} \vdash \varphi$, *then there are* $a_0, a_1 \in \omega$ *such that:*

- $\mathrm{HA} \vdash (\mathrm{a}_0 \Vdash \varphi)$
- $\mathrm{HA} \vdash (a_1 \Vdash_{tr} \varphi)$

Example: Kleene recursive realizability

Some applications:

- HA is consistent with Church's thesis:

$$\forall n \, \exists m \, \varphi(n, m) \to \exists e \, \forall n \, \varphi(n, \{e\}(n))$$

- HA is closed under Church's thesis rule:

$$\frac{\forall n \, \exists m \, \varphi(n, m)}{\exists e \, \forall n \, \varphi(n, \{e\}(n))}$$

- HA has the disjunction property: $\mathsf{HA} \vdash \varphi \vee \psi$ implies $\mathsf{HA} \vdash \varphi$ or $\mathsf{HA} \vdash \psi$
- HA has the existence property: $\mathsf{HA} \vdash \exists n \, \varphi(n)$ implies $\mathsf{HA} \vdash \varphi(n)$, for some $n \in \omega$.

Partial combinatory algebras

A partial algebra is a set A together with a partial function $(a, b) \mapsto a \cdot b$ from $A \times A$ to A.

Definition

A partial algebra A is a pca if there are elements (combinators) **k** and **s** such that:

- $(\mathbf{k} \cdot a) \cdot b \simeq a$;
- $(\mathbf{s} \cdot a) \cdot b \downarrow$ and $((\mathbf{s} \cdot a) \cdot b) \cdot c \simeq (a \cdot c) \cdot (b \cdot c)$.

Notation

ab for $a \cdot b$. abc for $(ab)c$ etcetera.

Partial combinatory algebras

Theorem

The are pairing $\mathbf{p}$ *and unpairig combinators* $\mathbf{p}_0, \mathbf{p}_1$ *such that:*

- $\mathbf{p}ab\downarrow$;
- $\mathbf{p}_0(\mathbf{p}ab) \simeq a$ *and* $\mathbf{p}_1(\mathbf{p}ab) \simeq b$.

Theorem (recursion theorem)

There is a fixed point combinator $\mathbf{f}$ *such that:*

- $\mathbf{f}a\downarrow$;
- $\mathbf{f}ab \simeq a(\mathbf{f}a)b$.

Partial combinatory algebras

Theorem

There is a map $n \mapsto \bar{n}$ from ω to A and combinators **succ**, **pred** *(successor and predecessor combinators),* **d** *(definition by cases combinator) such that*

$$\mathbf{succ}\,\bar{n} \simeq \overline{n+1}, \qquad \mathbf{pred}\,\overline{n+1} \simeq \bar{n},$$

$$\mathbf{d}\bar{n}\bar{m}ab \simeq \begin{cases} a & n = m; \\ b & n \neq m. \end{cases}$$

Remark

Use Curry numerals. However, any good representation of natural numbers works. For instance $n \mapsto n$ in the case of Kleene first algebra.

Partial combinatory algebras: examples

- Kleene first algebra: ω with $\{a\}(b)$
- Kleene second algebra: ω^ω with $f|g$
- Term models
- Plotkin-Scott graph model
- Scott's D_∞ model

Generic realizability

Generic interpretation of quantifiers:

- $a \Vdash \forall x\, \varphi(x)$ iff $a \Vdash \varphi(x)$ for every $x \in D$.
- $a \Vdash \exists x\, \varphi(x)$ iff $a \Vdash \varphi(x)$ for some $x \in D$.

Why?

Kreisel-Troelstra generic realizability

- Theory of species HAS (essentially second-order arithmetic with intuitionistic logic)
- Kleene first algebra

Definition

- $a \Vdash n \in X$ iff $(a, n) \in X$
- $a \Vdash \forall X \, \varphi$ iff $\forall X \, a \Vdash \varphi(X)$
- $a \Vdash \exists X \, \varphi$ iff $\exists X \, a \Vdash \varphi(X)$
- other clauses as in Kleene recursive realizability

Generic realizability for set theory

Kreisel-Troelstra-style generic realizability for set theory without extensionality:

- Friedman. *Some applications of Kleene's methods for intuitionistic systems* (1973)
- Beeson. *Foundations of constructive mathematics* (1985)

Generic realizability for set theory with extensionality:

- McCarthy. PhD thesis (1985)
- Rathjen and collaborators

What is intuitionistic set theory?

- Myhill's IZF
- Aczel's CZF
- ...

Intuitionistic set theory

IZF consists of:

- extensionality, pairing, union, infinity, set induction,
- separation: the set $\{z \in x \mid \varphi(z)\}$ exists for all formulas φ,
- collection: $\forall u \in x \, \exists v \, \varphi(u, v) \to \exists y \, \forall u \in x \, \exists v \in y \, \varphi(u, v)$, for all formulas φ,
- powerset.

Intuitionistic set theory

CZF consists of:

- extensionality, pairing, union, infinity, set induction,
- bounded separation: the set $\{z \in x \mid \varphi(z)\}$ exists for all bounded formulas φ,
- strong collection: $\forall u \in x\, \exists v\, \varphi(u, v) \to \exists y\, (\forall u \in x\, \exists v \in y\, \varphi(u, v) \wedge \forall v \in y\, \exists u \in x\, \varphi(u, v))$, for all formulas φ,
- subset collection: $\forall x\, \forall y\, \exists z\, \forall p\, (\forall u \in x\, \exists v \in y\, \varphi(u, v, p) \to \exists q \in z\, (\forall u \in x\, \exists v \in q\, \varphi(u, v, p) \wedge \forall v \in q\, \exists u \in x\, \varphi(u, v, p)))$, for all formulas φ.

Generic realizability for set theory

McCarty for IZF and Rathjen for CZF.

- A pca
- $\mathrm{V}(A)$ universe (domain of the intepretation)
- $a \Vdash \varphi$, for φ with parameters in $\mathrm{V}(A)$

Definition (Universe)

In IZF,

- $V(A)_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}(A \times V(A)_\beta)$
- $V(A) = \bigcup_\alpha V(A)_\alpha$

In CZF, $\mathrm{V}(A)$ is inductively defined by:

- if $x \subseteq A \times \mathrm{V}(A)$ (x consists of pairs $\langle a, y \rangle$ with $a \in A$ and $y \in \mathrm{V}(A)$), then $x \in \mathrm{V}(A)$

Generic realizability for set theory

Notation

$\langle x, y\rangle$ for set-theoretic pairing. a_i for $\mathbf{p}_i a$.

Definition (generic realizability)

Atomic case:

- $a \Vdash x \in y$ iff there is a z such that $\langle a_0, z\rangle \in y$ and $a_1 \Vdash x = z$
- $a \Vdash x = y$ iff $\langle b, z\rangle \in x$ implies $(ab)_0 \Vdash z \in y$ and $\langle b, z\rangle \in y$ implies $(ab)_1 \Vdash z \in x$.

Generic realizability for set theory

Connectives:

- as in Kleene recursive realizability

Unbounded quantifiers:

- $a \Vdash \forall x \varphi(x)$ iff $a \Vdash \varphi(x)$ for every $x \in \mathrm{V}(A)$.
- $a \Vdash \exists x \varphi(x)$ iff $a \Vdash \varphi(x)$ for some $x \in \mathrm{V}(A)$.

Bounded quantifiers (Rathjen):

- $a \Vdash \forall x \in y\, \varphi(x)$ iff $\langle b, x \rangle \in y$ implies $ab \Vdash \varphi(x)$
- $a \Vdash \exists x \in y\, \varphi(x)$ iff there exists x such that $\langle a_0, x \rangle \in y$ and $a_1 \Vdash \varphi(x)$

Theorem (McCarty, Rathjen)

Let T be IZF or CZF. For every theorem φ of T, there is a closed application term $\mathbf{e}$ *such that T proves* $\mathbf{e} \Vdash \varphi$.

Proof.

- Logical axioms and rules. Routine.
- Equality axioms. Recursion theorem.
- Extensionality. Recursion theorem.
- Pairing. Given $x, y \in \mathrm{V}(A)$, define $z = \{\langle \mathbf{0}, x\rangle, \langle \mathbf{1}, y\rangle\}$.
- Union. Given $x \in \mathrm{V}(A)$, define $y = \bigcup_{\langle a,u\rangle \in x} u$.

- Infinity. Let $\dot{\omega} = \{\langle \bar{n}, \dot{n} \rangle : n \in \omega\}$, where $\dot{n} = \{\langle \bar{m}, \dot{m} \rangle : m \in n\}$.
- Induction. Recursion theorem.
- Separation. Given x, let
  $y = \{\langle \mathbf{p}ab, u \rangle \mid \langle a, u \rangle \in x \wedge b \Vdash \varphi(u)\}$.
  Fine for IZF.
  Bounded separation for CZF follows by general results.
  Essentially one can still define sets by bounded comprehension in a language extended with function symbols for definable functions.
  $f_{\in}(x, y) = \{a \in A \mid a \Vdash x \in y\}$: $a \Vdash x \in y$ iff $a \in f_{\in}(x, y)$.
  $f_{=}(x, y) = \{a \in A \mid a \Vdash x = y\}$: $a \Vdash x = y$ iff $a \in f_{=}(x, y)$.
- Routine

Generic realizability with truth

Goal: define universe $\mathrm{V}(A)$ such that every set in V has a name in $\mathrm{V}(A)$, and

$$(a \Vdash_{tr} \varphi) \to \varphi^{\circ},$$

where $x \mapsto x^{\circ}$ is the evaluation map from $\mathrm{V}(A)$ to V.

Generic realizability with truth

Rathjen.

Definition (Universe)

In CZF, $\mathrm{V}(A)$ is inductively defined by the following clause:

- if $x^\circ \in \mathrm{V}$, $x^* \in \mathcal{P}(A \times \mathrm{V}(A))$, and for every $\langle a, \langle u^\circ, u^* \rangle\rangle \in x^*$ we have $u^\circ \in x^\circ$, then $\langle x^\circ, x^* \rangle \in \mathrm{V}(A)$

If x is the pair $\langle x_0, x_1 \rangle$, let

- $x^\circ = x_0$
- $x^* = x_1$

The intuition is that $\langle x^\circ, x^* \rangle \in \mathrm{V}(A)$ is a name for $x^\circ \in \mathrm{V}$. Note that

$$\{u^\circ \mid \exists a \in A \, \langle a, u \rangle \in x^*\} \subseteq x^\circ$$

Generic realizability with truth

In each clause of $a \Vdash \varphi$ add φ°.

Definition (generic realizability with truth)

Atomic case:

- $a \Vdash_{tr} x \in y$ iff $x^\circ \in y^\circ$ and there exists z such that $\langle a_0, z \rangle \in y^*$ and $a_1 \Vdash x = z$
- $a \Vdash_{tr} x = y$ iff $x^\circ = y^\circ$, $\langle b, z \rangle \in x^*$ implies $(ab)_0 \Vdash z \in y)$ and $\langle b, z \rangle \in y^*$ implies $(ab)_1 \Vdash z \in x$

Connectives:

- $\wedge$ and $\vee$ as in generic realizability
- $a \Vdash_{tr} \neg\varphi$ iff $\neg\varphi^\circ$
- $a \Vdash \varphi \to \psi$ iff $\varphi^\circ \to \psi^\circ$ and $b \Vdash_{tr} \varphi$ implies $ab \Vdash_{tr} \psi$

Generic realizability with truth

Unbounded quantifiers:

- as in generic realizability

Bounded quantifiers:

- $a \Vdash_{tr} \forall x \in y\, \varphi$ iff $\forall x \in y^\circ\, \varphi^\circ$ and $\langle b, x \rangle \in y^*$ implies $ab \Vdash_{tr} \varphi$
- $a \Vdash_{tr} \exists x \in y\, \varphi$ iff there exists x such that $\langle a_0, x \rangle \in y^*$ and $a_1 \Vdash_{tr} \varphi$

Applications of generic realizability

Let T be CZF or IZF.

- T has the disjunction property ($T \vdash \varphi \vee \psi$ then $T \vdash \varphi$ or $T \vdash \psi$) and the numerical existence property ($T \vdash \exists n\, \varphi(n)$ then $T \vdash \varphi(n)$, for some standard n).
- T is consistent with Church's thesis and closed under Church's rule.
- T is closed under the uniformity rule UZR: if $T \vdash \forall x\, (\varphi(x) \vee \psi(x))$, then either $T \vdash \forall x\, \varphi(x)$ or $T \vdash \forall x\, \psi(x)$
- CZF does not have the existence property, which says that whenever $T \vdash \exists x\, \varphi(x)$, then there is a formula $\vartheta(x)$ such that $T \vdash \exists! x\, (\vartheta(x) \wedge \varphi(x))$.

New applications

- Joint work with Michael Rathjen: *Extensional realizability for intuitionistic set theory*, JLC 2020
- Work in progress with Takako Nemoto and Michael Rathjen

Choice in intuitionistic set theory

Known results:

- $\mathrm{IZF} + \mathrm{AC}_{FT}$ is Π^0_2 conservative over IZF (Friedman)
- Same for CZF (Rathjen)
- $\mathrm{IZF} + \mathrm{DC}_{FT} + \mathrm{AC}_{0,\tau}$ is conservative over IZF for arithmetical sentences (Friedman and Scedrov, Beeson)
- $S + \mathrm{AC}_{FT}$ is conservative over S for arithmetical sentences, for various subtheories of CZF (Goordev)

The proof of the third item uses Kreisel-Troelstra-style generic realizability with Kleene first algebra: wrong!

Extensional generic realizability

We introduce a notion of extensional generic realizability

$$a = b \Vdash \varphi$$

that works with any pca.

Theorem (Frittaion and Rathjen)

Let T be IZF or CZF. Then $T + \mathrm{AC}_{FT}$ is interpretable in T under extensional generic realizability, and in particular is conservative over T for Π^0_2 sentences.

By combining extensional generic realizability (using Kleene first algebra) with forcing (as in Goodman-Beeson), one can show arithmetic conservativity, not just Π^0_2.

Choice in all finite types

Definition

Finite types σ are defined by clauses:

- $0 \in \mathrm{FT}$;
- if $\sigma, \tau \in \mathrm{FT}$, then $\sigma\tau \in \mathrm{FT}$;

Extensions:

- $F_0 = \omega$;
- $F_{\sigma\tau} = F_\sigma \to F_\tau = \{\text{total functions from } F_\sigma \text{ to } F_\tau\}$.

Definition

Finite type AC_{FT} consists of formulas

$$\forall x^\sigma\, \exists y^\tau\, \varphi(x, y) \to \exists f^{\sigma\tau}\, \forall x^\sigma\, \varphi(x, f(x)), \qquad (\mathrm{AC}_{\sigma,\tau})$$

for all σ and τ.

Extensional generic realizability

Generic realizability does not do the job.

- $\mathrm{AC}_{0,\tau}$ for $\tau \in \{0,1\}$ holds in $\mathrm{V}(A)$ for any partial combinatory algebra A.
- $\mathrm{AC}_{1,\tau}$ for $\tau \in \{0,1\}$ holds in $\mathrm{V}(A)$ by taking, e.g., Kleene's second algebra.
- That's all

Extensional generic realizability

What does it mean to realize $\mathsf{AC}_{\sigma,\tau}$?

Let $\vartheta_\sigma(z)$ define F_σ:

- $Qx^\sigma \ldots$ stands for $\forall z\,(\vartheta_\sigma(z) \to Qx \in z \ldots)$

Challenge: find names $\dot{F}_\sigma$ and realizers for:

- $\vartheta_0(\dot{F}_0)$
- $\forall f\,(f \in \dot{F}_{\sigma\tau} \leftrightarrow f \in \mathrm{Func}(\dot{F}_\sigma, \dot{F}_\tau))$
- $\forall x \in \dot{F}_\sigma\, \exists y \in \dot{F}_\tau\, \varphi(x,y) \to \exists f \in \mathrm{Func}(\dot{F}_\sigma, \dot{F}_\tau)\, \forall x \in \dot{F}_\sigma\, \varphi(x, f(x))$

Extensional generic realizability

The plan is to build HEO (Hereditarily Effective Operations) and define names $\dot{F}_\sigma$ out of A.

- $a \sim_0 b$ iff $a = b = \bar{n}$ for $n \in \omega$
- $a \sim_{\sigma\tau} b$ iff $c \sim_\sigma d$ implies $ac \sim_\tau bd$.

We say that $a \in A$ has type σ if $a \sim_\sigma a$.

Extensional generic realizability

Extensional realizability $a = b \Vdash \varphi$.

- We know what to do with connectives and quantifiers
- We are left with atomic formulas.

Definition (Universe)

Let $\mathrm{V}_{\mathrm{ex}}(A)$ consist of sets of triples $\langle a, b, x\rangle$ with $a, b \in A$ and $x \in \mathrm{V}_{\mathrm{ex}}(A)$.

Extensional generic realizability

Definition (extensional generic realizability)

Atomic case:

- $a = b \Vdash x \in y$ iff $\exists z\, (\langle a_0, b_0, z\rangle \in y \wedge a_1 = b_1 \Vdash x = z)$
- $a = b \Vdash x = y$ iff $\forall \langle c, d, z\rangle \in x\, ((ac)_0 = (bd)_0 \Vdash z \in y)$ and $\forall \langle c, d, z\rangle \in y\, ((ac)_1 = (bd)_1 \Vdash z \in x)$

Connectives:

- $a = b \Vdash \varphi \wedge \psi$ iff $a_0 = b_0 \Vdash \varphi \wedge a_1 = b_1 \Vdash \psi$
- $a = b \Vdash \varphi \vee \psi$ iff $a_0 = b_0 = \mathbf{0} \wedge a_1 = b_1 \Vdash \varphi$ or $a_0 = b_0 = \mathbf{1} \wedge a_1 = b_1 \Vdash \psi$
- $a = b \Vdash \neg\varphi$ iff $\forall c, d\, \neg(c = d \Vdash \varphi)$
- $a = b \Vdash \varphi \rightarrow \psi$ iff $\forall c, d\, (c = d \Vdash \varphi \rightarrow ac = bd \Vdash \psi)$

Extensional generic realizability

Unbounded quantifiers:

- $a = b \Vdash \forall x\, \varphi$ iff $a = b \Vdash \varphi$ for every $x \in \mathrm{V}_{\mathrm{ex}}(A)$
- $a = b \Vdash \exists x\, \varphi$ iff $a = b \Vdash \varphi$ for some $x \in \mathrm{V}_{\mathrm{ex}}(A)$

Bounded quantifiers:

- $a = b \Vdash \forall x \in y\, \varphi$ iff $\forall \langle c, d, x \rangle \in y\, (ac = bd \Vdash \varphi)$
- $a = b \Vdash \exists x \in y\, \varphi$ iff $\exists x\, (\langle a_0, b_0, x \rangle \in y \wedge a_1 = b_1 \Vdash \varphi)$

$a \Vdash \varphi$ means $a = a \Vdash \varphi$.

Extensional generic realizability

Definition

- $\dot{F}_\sigma = \{\langle a, b, a^\sigma \rangle \mid a \sim_\sigma b\}$
- if $a = \bar{n}$, then $a^0 = \{\langle \bar{m}, \bar{m}, b^0 \rangle \mid b = \bar{m} \wedge m < n\}$
- if $a \sim_{\sigma\tau} a$, then $a^{\sigma\tau} = \{\langle c, d, \langle c^\sigma, e^\tau \rangle_A \rangle \mid c \sim_\sigma d \wedge ac \simeq e\}$

Theorem

For all finite types σ and τ there exists a closed application term **c** *such that* CZF *proves*

$$\mathbf{c} \Vdash \forall x^\sigma \, \exists y^\tau \, \varphi(x, y) \to \exists f^{\sigma\tau} \, \forall x^\sigma \, \varphi(x, f(x)).$$

Finite types rules in intuitionistic set theory

Question

Is IZF *(*CZF*) closed under the following rules?*

The rule of choice $\mathrm{CR_{FT}}$ in finite types

$$\frac{\forall x^\sigma\,\exists y^\tau\,\varphi(x,y)}{\exists f^{\sigma\tau}\,\forall x^\sigma\,\varphi(x,f(x))} \qquad (\mathrm{CR}_{\sigma,\tau})$$

Forms of independence of premise rule $\mathrm{IPR_{FT}}$ in finite types

$$\frac{\varphi(x)\to\exists y^\sigma\,\psi(x,y)}{\exists y^\sigma\,(\varphi(x)\to\psi(x,y))} \qquad (\mathrm{IPR}_\sigma)$$

Applying generic realizability with truth

- Build a pca A on top of $\mathbb{F} = \bigcup_\sigma F_\sigma$
- Use generic realizability with truth

Definition (pca over $\mathbb{F}$)

A pca over $\mathbb{F}$ is a pca A such that:

- $\mathbb{F} \subseteq A$;
- $f \cdot x \simeq f(x)$ for $f \in F_{\sigma\tau}$ and $x \in F_\sigma$.

More in general, there is a monomorphism (of partial algebras) from $\mathbb{F}$ to A, that is, an injective function $x \mapsto \bar{x}$ from $\mathbb{F}$ to A such that $\bar{f} \cdot \bar{x} \simeq \overline{f(x)}$ for $f \in F_{\sigma\tau}$ and $x \in F_\sigma$.

Applying generic realizability with truth

Challenge: find names $\dot{F}_\sigma$ and truth realizers for:

- $\vartheta_0(\dot{F}_0)$
- $\forall f\,(f \in \dot{F}_{\sigma\tau} \leftrightarrow f \in \mathrm{Func}(\dot{F}_\sigma, \dot{F}_\tau))$
- $\forall x \in \dot{F}_\sigma\, \exists y \in \dot{F}_\tau\, \varphi(x,y) \to \exists f \in \mathrm{Func}(\dot{F}_\sigma, \dot{F}_\tau)\, \forall x \in \dot{F}_\sigma\, \varphi(x, f(x))$

We must have:

$$\dot{F}_\sigma = \langle F_\sigma, E_\sigma \rangle$$

Remember: $\langle x^\circ, x^* \rangle \in \mathrm{V}(A)$ is a name for $x^\circ \in \mathrm{V}$

Applying generic realizability with truth

Definition (Canonical names for objects of finite type and extensions)

Let A be a pca over $\mathbb{F}$. Let

$$\dot{F}_\sigma = \langle F_\sigma, \{\langle \bar{x}, \dot{x} \rangle \mid x \in F_\sigma\} \rangle,$$

where

$$\dot{n} = \langle n, \{\langle \bar{m}, \dot{m} \rangle \mid m < n\} \rangle,$$

and for $f \in F_{\sigma\tau}$,

$$\dot{f} = \langle f, \{\langle \bar{x}, \langle \dot{x}, \dot{y} \rangle_A \rangle \mid x \in F_\sigma \wedge f(x) = y\} \rangle.$$

Applying generic realizability with truth

Definition (CZF)

A name $x = \langle x^\circ, x^* \rangle \in \mathrm{V}(A)$ is bijectively presented if

(i) $x^\circ = \{u^\circ \mid \exists a\,(\langle a, u\rangle \in x^*)\}$;

(ii) if $\langle a, u\rangle, \langle b, v\rangle \in x^*$, then $a = b$ iff $u^\circ = v^\circ$.

In other words,

$$\{\langle a, u^\circ\rangle \mid \langle a, u\rangle \in x^*\} \colon A \rightharpoonup x^\circ$$

is a one-to-one function onto x°.

Lemma

Every $\dot{x}$ with $x \in \mathbb{F}$ and every $\dot{F}_\sigma$ is bijectively presented.

Applying generic realizability with truth

Theorem (choice for bijectively presented names)

CZF *proves*

$$\exists a\,(a \Vdash_{tr} \forall u \in x\, \exists v \in y\, \varphi(x, y)) \rightarrow \exists f : x^\circ \rightarrow y^\circ\, \forall u \in x^\circ\, \varphi^\circ(u, f(u)),$$

for all bijectively presented names $x, y \in \mathrm{V}(A)$.

We are fine if we find truth realizers for $\vartheta_\sigma(\dot{F}_\sigma)$.

Introducing extensive pca's over $\mathbb{F}$

Definition (extensive pca over $\mathbb{F}$)

A pca A over $\mathbb{F}$ is *extensive* on $\mathbb{F}$ if for all σ and τ there is a combinator (the $\sigma\tau$ combinator) $\mathbf{i}_{\sigma\tau}$ such that

$$\mathbf{i}_{\sigma\tau} \cdot a \simeq \bar{f},$$

if $f = \{\langle x, y\rangle \mid x \in F_\sigma \wedge y \in F_\tau \wedge a \cdot \bar{x} \simeq \bar{y}\} \in F_{\sigma\tau}$.

Theorem

Let A be an extensive pca over $\mathbb{F}$ in CZF. *Then for every type σ there exists a closed application term* $\mathbf{c}$ *such that* CZF *proves* $\mathbf{c} \Vdash_{tr} \vartheta_\sigma(\dot{F}_\sigma)$.

Let T be IZF or CZF.

Theorem

T is closed under $\mathrm{CR}_{\mathrm{FT}}$.

Theorem

T is closed under

$$\frac{\forall x\, \exists y^{\sigma}\, \varphi(x, y)}{\exists y^{\sigma}\, \forall x\, \varphi(x, y)} \qquad (\mathrm{UR}_{\sigma})$$

Proof. Use an extensive pca over $\mathbb{F}$ in T.

Independence of premise rules

Two kinds of rule:

$$\frac{\varphi(x) \to \exists y^\sigma\, \psi(x,y)}{\exists y^\sigma\, (\varphi(x) \to \psi(x,y))}$$

vs

$$\frac{\varphi(x) \to \exists y^\sigma\, \psi(x,y)}{\exists y\, (\varphi(x) \to y \in F_\sigma \wedge \psi(x,y))}$$

Independence of premise rules

Theorem (some items are still conjectures)

Let T be IZF *or* CZF*. Then T is closed under the following independence of premise rules:*

$$\frac{\forall x\,(\neg\varphi(x) \rightarrow \exists y^{\sigma}\,\psi(x,y))}{\exists y\,\forall x\,(\neg\varphi(x) \rightarrow y \in F_{\sigma} \wedge \psi(x,y))} \tag{1}$$

$$\frac{\forall x\,(\neg\varphi(x) \rightarrow \exists y^{\sigma}\,\psi(x,y)) \qquad \exists x\,\neg\varphi(x)}{\exists y^{\sigma}\,\forall x\,(\neg\varphi(x) \rightarrow \psi(x,y))} \tag{2}$$

Independence of premise rules

$$\frac{\forall x\,(\forall z\,\theta(x,z) \to \exists y^{\sigma}\,\psi(x,y)) \qquad \forall x\,\forall z\,(\theta(x,z) \vee \neg\theta(x,z))}{\exists y^{\sigma}\,\forall x\,(\forall z\,\theta(x,z) \to \psi(x,y))} \tag{3}$$

$$\frac{\forall x\,(\forall z^{\rho}\,\theta(x,z) \to \exists y^{\sigma}\,\psi(x,y)) \qquad \forall x\,\forall z^{\rho}\,(\theta(x,z) \vee \neg\theta(x,z))}{\exists y\,\forall x\,(\forall z^{\rho}\,\theta(x,z) \to y \in F_{\sigma} \wedge \psi(x,y))} \tag{4}$$

Proof. For (1) and (4) use a total extensive pca over $\mathbb{F}$.

Question

- *Is there an extensive pca over $\mathbb{F}$. Can we prove it in* CZF*?*
- *Is there a total extensive pca over $\mathbb{F}$. Can we prove it in* CZF*?*

- YES YES
- We can prove that there is a total pca over $\mathbb{F}$ in CZF by adapting the graph model construction. Not extensive though.

Thank you!