The fixed-point property for represented spaces

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Fixed-point property

- Many mathematical fields have their fixed-point theorems: topology, order theory, convex analysis, etc.
- In computability theory: Kleene's Recursion Theorem, its extension by Ershov to numbered sets,
- In computable analysis: Kreitz, Weihrauch, 1985

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Computability theory	Computable analysis
Numbered sets	Represented spaces
Computable multi-valued functions	Continuous multi-valued functions

From numbered sets to represented spaces

Some results easily extend:

- Ershov: A total numbering satisfies the 2nd recursion theorem $\iff$ it is precomplete,
- Weihrauch: Effective domains satisfy the 2nd recursion theorem.

From numbered sets to represented spaces

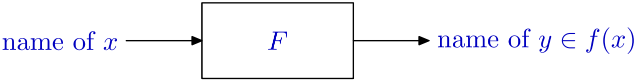
Some results become possible: continuity is smoother than computability.

Problems

- Give characterizations of classes of spaces with the FPP,
- Why does the FPP usually hold *uniformly*?
- Is the diagonal argument the only way to prove the FPP?

Represented spaces

- Baire space: $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$,
- Represented space: pair $\mathbf{X} = (X, \delta_X)$, where $\delta_X :\subseteq \mathcal{N} \to X$ is surjective,
- A multifunction $f : \mathbf{X} \rightrightarrows \mathbf{Y}$ is computable if it has a computable realizer $F :\subseteq \mathcal{N} \to \mathcal{N}$:



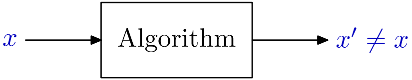
- f is continuous if it has a continuous realizer.

The fixed-point property

Definition

A represented space $\mathbf{X}$ has the **fixed-point property (FPP)** if every continuous multifunction $h : \mathbf{X} \rightrightarrows \mathbf{X}$ has a fixed-point, i.e. some $x \in \mathbf{X}$ such that $x \in h(x)$.

Computable fixed-point free multifunction:



Examples

Which spaces have the fixed-point property?

- $\mathbb{R}$
- $[0, 1]$
- $[0, 1]_<$
- $(0, 1]_<$
- $[0, 1)_<$
- $\mathcal{P}(\omega)$
- $\utilde{\mathbf{\Sigma}}^0_n(\mathcal{N})$
- $\utilde{\mathbf{\Delta}}^0_n(\mathcal{N})$

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- $\mathbb{R}$ **NO**: $h(x) = x + 1$
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- $\mathcal{P}(\omega)$ **YES**
- $\utilde{\mathbf{\Sigma}}^0_n(\mathcal{N})$ **YES**
- $\utilde{\mathbf{\Delta}}^0_n(\mathcal{N})$ **NO**: $h(A) = A^c$

Diagonal argument

The spaces $[0,1]_<$, $\mathcal{P}(\omega)$ and $\underset{\sim}{\mathbf{\Sigma}}^0_n(\mathcal{N})$ have the FPP.

Diagonal argument

If there is a continuous surjection $\phi : \mathcal{N} \to \mathscr{C}(\mathcal{N}, \mathbf{X})$, then $\mathbf{X}$ has the FPP.

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Proof.

- Given $h : \mathbf{X} \rightrightarrows \mathbf{X}$,
- Let $f \in \mathscr{C}(\mathcal{N}, \mathbf{X})$ be such that $f(p) \in h(\phi(p)(p))$,
- One has $f = \phi(p_0)$ for some p_0,
- $\phi(p_0)(p_0)$ is a fixed-point of h. □

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Key fact

Every continuous multifunction from $\mathcal{N}$ has a continuous single-valued selector.

Least element

Let τ be the final topology of the representation. Let $\leq$ be the specialization preorder:

$$x \leq y \iff \text{every neighborhood of } x \text{ contains } y.$$

Proposition

If $\mathbf{X}$ has the fixed-point property, then $\mathbf{X}$ has a least element.

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$$x \leq y \iff \text{every neighborhood of } x \text{ contains } y.$$

Proposition

If $\mathbf{X}$ *has the fixed-point property, then* $\mathbf{X}$ *has a least element.*

Proof.

$\mathbf{X}$ has no least element $\iff$ there exists a proper open cover $(U_i)_{i\in\mathbb{N}}$:

- $X = \bigcup_i U_i$,
- $X \neq U_i$ for each i.

□

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- $X = \bigcup_i U_i$,
- $X \neq U_i$ for each i.

We build a fixed-point free multifunction $h : \mathbf{X} \rightrightarrows \mathbf{X}$.
Given x, find i such that $x \in U_i$, then output some $x' \notin U_i$. □

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Proposition

If $\mathbf{X}$ has the fixed-point property, then $\mathbf{X}$ has a least element.

Therefore, the final topology is:

- Compact,
- Not T_1 (unless $\mathbf{X}$ is a singleton).

Least element

	FPP?	Least element?
• $\mathbb{R}$	**NO**	**NO**
• $[0,1]$	**NO**	**NO**
• $[0,1]_<$	**YES**	**YES**: 0
• $(0,1]_<$	**NO**	**NO**
• $[0,1)_<$	**NO**	**YES**: 0
• $\mathcal{P}(\omega)$	**YES**	**YES**: $\emptyset$
• $\boldsymbol{\Sigma}^0_n(\mathcal{N})$	**YES**	**YES**: every element
• $\boldsymbol{\Delta}^0_n(\mathcal{N})$	**NO**	**YES**: every element

Uniform fixed-point property

The spaces $\mathcal{P}(\omega)$, $[0,1]_<$, $\underset{\sim}{\mathbf{\Sigma}}^0_n(\mathcal{N})$ have the fixed-point property.

Moreover, a fixed-point for $h : \mathbf{X} \rightrightarrows \mathbf{X}$ can be *uniformly* computed from h.

Uniform fixed-point property

- A uniform fixed-point property is defined in **[Kreitz, Weihrauch, 85]**: "satisfying the t-*recursion theorem*",
- Too weak: does not imply the fixed-point property.

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Definition (Attempt)

A represented space $\mathbf{X}$ has the **uniform fixed-point property (UFPP)** if given $H :\subseteq \mathcal{N} \to \mathcal{N}$, one can continuously find some $p \in \mathcal{N}$ such that

$$\delta_X(p) = \delta_X \circ H(p).$$

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Definition

A represented space $\mathbf{X}$ has the **uniform fixed-point property (UFPP)** if given $H :\subseteq \mathcal{N} \to \mathcal{N}$, one can continuously find some $p \in \mathrm{dom}(\delta_{\mathbf{X}})$ such that

$$x \in \mathrm{dom}(\delta_X \circ H) \implies \delta_X(p) = \delta_X \circ H(p).$$

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[Kreitz, Weihrauch, 85] assumes H is total, and does not require $p \in \mathrm{dom}(\delta_X)$.

Uniform fixed-point property

Theorem

$\mathbf{X}$ *has the uniform fixed-point property*

$$\Longleftrightarrow$$

Every partial continuous function $f :\subseteq \mathcal{N} \to \mathbf{X}$ *has a total continuous extension* $\tilde{f} : \mathcal{N} \to \mathbf{X}$.

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Every partial continuous function $f :\subseteq \mathcal{N} \to \mathbf{X}$ *has a total continuous extension* $\tilde{f} : \mathcal{N} \to \mathbf{X}$.

- This property is called **multi-retraceability** in **[Brattka, Gherardi, 2021]**
- It is equivalent to the **effective discontinuity**, defined in **[Brattka, 2020]**, of the multifunction $h(x) = X \setminus \{x\}$.

Uniform fixed-point property

Therefore, from the results in **[Brattka, 2020]**:

Corollary

Assuming the Axiom of Determinacy (AD),

$$\mathit{FPP} \iff \mathit{UFPP}.$$

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$$FPP \iff UFPP.$$

Proof idea.

Game: Players I and II play $x_1, x_2 \in \mathbf{X}$. Player II wins if $x_2 \neq x_1$.

- A winning strategy for Player II is a fixed-point free continuous multifunction.
- A winning strategy for Player I witnesses the uniform fixed-point property. □

Uniform fixed-point property

Therefore, from the results in **[Brattka, 2020]**:

Corollary

Assuming the Axiom of Determinacy (AD),

$$\mathit{FPP} \iff \mathit{UFPP}.$$

- Holds for most natural spaces without (AD),
- We will see classes of spaces for which (AD) can be dropped.

Uniform fixed-point property

Theorem

Assuming the Axiom of Choice,

$$\text{FPP} \not\Longleftrightarrow \text{UFPP}.$$

Proof.

Let $X = \{0,1\}$, $A \subseteq \mathcal{N}$ and $\delta = \mathbf{1}_A$.
(X, δ) has the FPP $\iff A \not\leq_{\text{Wadge}} A^c$.
Build A by transfinite induction (similar to the construction of a Bernstein set). □

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Open problem

Build an *admissibly* represented space satisfying the FPP, but not the UFPP.

Diagonal argument

Reminder

The spaces $\mathcal{P}(\omega)$, $[0,1]_<$, $\utilde{\mathbf{\Sigma}}^0_n(\mathcal{N})$ have the fixed-point property.
Proved using the diagonal argument.

Question

Is the diagonal argument the only way to prove the FPP?

Diagonal argument

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Is the diagonal argument the only way to prove the FPP?

There is a continuous surjection $\phi : \mathcal{N} \to \mathscr{C}(\mathcal{N}, \mathbf{X})$

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$\mathbf{X}$ has the FPP.

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Is the diagonal argument the only way to prove the FPP?

Assuming (AD),

$$\text{There is a continuous surjection } \phi : \mathcal{N} \to \mathscr{C}(\mathcal{N}, \mathbf{X}) \iff \mathbf{X} \text{ has the FPP.}$$

Complexity of equality

If $A \subseteq \mathbf{X}$ has the FPP/UFPP, then A is no more complex than equality on $\mathbf{X}$.

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Definition (Descriptive complexity)

In a represented space $\mathbf{X} = (X, \delta)$ with δ total,

$$A \in \Gamma(\mathbf{X}) \iff \delta^{-1}(A) \in \Gamma(\mathcal{N}),$$

$$A \subseteq \mathbf{X} \text{ is } \Gamma\text{-hard} \iff \delta^{-1}(A) \text{ is } \Gamma\text{-hard}.$$

Complexity of equality

Let $\Gamma \in \{\underset{\sim}{\boldsymbol{\Sigma}}^0_\alpha, \underset{\sim}{\boldsymbol{\Pi}}^0_\alpha, \underset{\sim}{\boldsymbol{\Sigma}}^1_\alpha, \underset{\sim}{\boldsymbol{\Pi}}^1_\alpha\}$ with α a countable ordinal.

Theorem

Assume that equality on $\mathbf{X}$ belongs to $\Gamma(\mathbf{X} \times \mathbf{X})$ and let $A \subseteq X$:

- *If A has the UFPP then $A \in \Gamma(\mathbf{X})$,*
- *If A has the FPP then A is not $\check{\Gamma}$-hard.*

$$\begin{array}{ccc} \text{UFPP} & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \text{FPP} & \longrightarrow & \text{not } \check{\Gamma}\text{-hard} \end{array}$$

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And assuming (AD),

$$\begin{array}{ccc} \text{UFPP} & \longrightarrow & \Gamma \\ \updownarrow & & \updownarrow \\ \text{FPP} & \longrightarrow & \text{not } \check{\Gamma}\text{-hard} \end{array}$$

Example

In $\mathcal{P}(\omega)$ equality belongs to $\utilde{\boldsymbol{\Pi}}^0_2$, so for $A \subseteq \mathcal{P}(\omega)$,

$$\begin{array}{ccc} \text{UFPP} & \longrightarrow & \utilde{\boldsymbol{\Pi}}^0_2 \\ \downarrow & & \downarrow \\ \text{FPP} & \longrightarrow & \text{not } \utilde{\boldsymbol{\Sigma}}^0_2\text{-hard} \end{array}$$

Remark

The $\utilde{\boldsymbol{\Pi}}^0_2$-subspaces of $\mathcal{P}(\omega)$ are the quasiPolish spaces **[de Brecht 2013]**.

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Remark

The $\boldsymbol{\Pi}^0_2$-subspaces of $\mathcal{P}(\omega)$ are the quasiPolish spaces **[de Brecht 2013]**.

In $\mathcal{P}(\omega)$, we will later prove UFPP $\longleftrightarrow$ FPP, *without assuming* (AD).

Subspace

Let $\mathbf{X}$ have the UFPP.

Proposition

A subspace $\mathbf{Y} \subseteq \mathbf{X}$ *has the UFPP* $\iff$ $\mathbf{Y}$ *is a multi-valued retract of* $\mathbf{X}$.

There exist $r : \mathbf{X} \rightrightarrows \mathbf{Y}$ and $s : \mathbf{Y} \to \mathbf{X}$ such that $r \circ s = \mathrm{id}_{\mathbf{Y}}$.

Example

A countably-based T_0-space has the UFPP $\iff$ it is a multi-valued retract of $\mathcal{P}(\omega)$.

FPP

UFPP

Classes of spaces
- Countably-based spaces
- Spaces of open sets

Base-complexity

Countably-based T_0-spaces

Let $\mathbf{X}$ be a countably-based T_0-space with the standard representation.

Theorem

The following statements are equivalent:

1. $\mathbf{X}$ *has the fixed-point property,*
2. $\mathbf{X}$ *has the uniform fixed-point property,*
3. $\mathbf{X}$ *is a multi-valued retract of* $\mathcal{P}(\omega)$*,*
4. $\mathbf{X}$ *is a pointed* ω*-continuous dcpo with the Scott topology.*

We do not assume (AD).

Let's show why FPP $\implies$ ω-continuous dcpo.

Proof ideas

Let us illustrate why, for subsets of $(\mathcal{P}(\omega), \subseteq)$:

- Not a dcpo $\implies$ fixed-point free multifunction,
- Not ω-continuous $\implies$ fixed-point free multifunction.

Proof ideas: dcpo

The set $\mathbf{X} = \mathcal{P}(\omega) \setminus \{\omega\}$ admits a fixed-point free continuous function $h : \mathbf{X} \to \mathbf{X}$:

$$h(A) = \{0, \ldots, n\}, \text{ where } n \notin A \text{ is minimal.}$$

Proof ideas: dcpo

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What's going on?

We are exploiting that $\mathbf{X}$ is not a dcpo: the set

$$D = \{\{0, \ldots, n\} : n \in \omega\}$$

is directed but has no sup.

Proof ideas: ω-continuity

The set $\mathbf{X} = \{\emptyset\} \cup \{A \subseteq \omega : A \text{ is infinite}\}$ has a fixed-point free continuous multifunction:

- Given $A \in \mathbf{X}$, we start producing ω,
- If we detect that $A \neq \emptyset$, then we pause and find some $n \in A$ that we do not have enumerated yet,
- We then produce $\omega \setminus \{n\}$.

Proof ideas: ω-continuity

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What's going on?

- Say that $A \ll_t B$ if $B \in \text{int}(\uparrow A)$.
- ω-continuity means that $\ll_t$ is rich: $B = \sup\{A : A \ll_t B\}$.
- $\mathbf{X}$ is not ω-continuous: if $A \neq \emptyset$, then $A \not\ll_t \omega$.

Countably-based spaces

- Multifunctions are much more flexible than functions,
- The *single-valued* FPP is much harder to understand, even for finite spaces.

For finite T_0-spaces,

- FPP $\iff$ It has a least element,
- Single-valued FPP $\iff$ Single-valued FPP for finite posets, which is an open problem.

Spaces of open sets

Let $\mathbf{X}$ be admissibly represented. $\mathcal{O}(\mathbf{X})$ has an admissible representation.

Theorem

The following statements are equivalent:

- $\mathbf{X}$ *is countably-based,*
- $\mathcal{O}(\mathbf{X})$ *has the fixed-point property,*
- $\mathcal{O}(\mathbf{X})$ *has the uniform fixed-point property.*

Knaster-Tarski or Kleene's fixed-point theorems imply that continuous functions $\mathcal{O}(\mathbf{X}) \to \mathcal{O}(\mathbf{X})$ *always* have fixed-points.

Spaces of open sets: proof idea

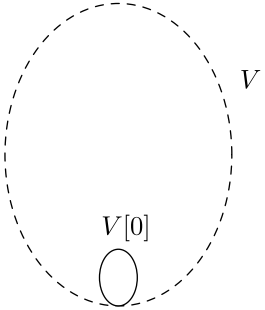
- In a countably-based space, enumerating an open set V means producing a growing sequence of *open* sets $V[s]$ such that $V = \bigcup_s V[s]$,
- When the space is not countably-based, the sets $V[s]$ are not always open.

For simplicity, let's work in a space where each $V[s]$ has empty interior.

Spaces of open sets: proof idea

Opponent gives some $U \in \mathcal{O}(\mathbf{X})$, we produce some $V \neq U$.

- Start enumerating some $V \neq \emptyset$,



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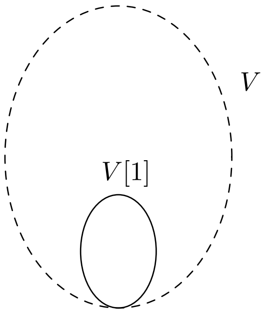

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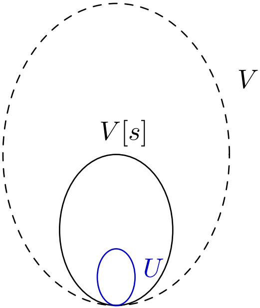

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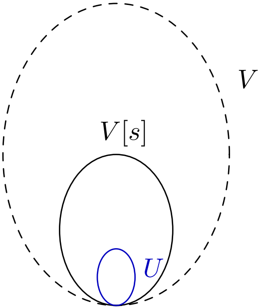
- Start enumerating some $V \neq \emptyset$,
- If we detect that $U \neq \emptyset$, then we pause,
- $V[s]$ has empty interior, so $U \not\subseteq V[s]$,



Spaces of open sets: proof idea

Opponent gives some $U \in \mathcal{O}(\mathbf{X})$, we produce some $V \neq U$.

- Start enumerating some $V \neq \emptyset$,
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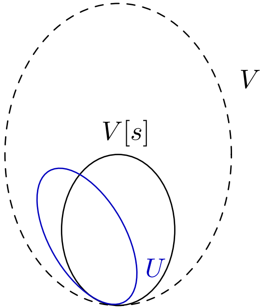
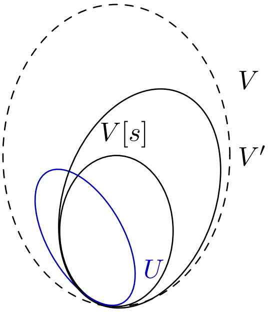


Spaces of open sets: proof idea

Opponent gives some $U \in \mathcal{O}(\mathbf{X})$, we produce some $V \neq U$.

- Start enumerating some $V \neq \emptyset$,
- If we detect that $U \neq \emptyset$, then we pause,
- $V[s]$ has empty interior, so $U \nsubseteq V[s]$,
- Produce some $V' \supseteq V[s]$ such that $U \nsubseteq V'$.



Base-complexity

Definition ([de Brecht, Schröder, Selivanov, 2016])

A topological space $\mathbf{X}$ is $\mathbf{Y}$-based if there is a continuous indexing $\mathbf{Y} \to \mathcal{O}(\mathbf{X})$ of a basis.

A hierarchy can be obtained by using the Kleene-Kreisel spaces $\mathbf{Y} = \mathbb{N}\langle\alpha\rangle$:

- $\mathbb{N}\langle 0\rangle = \mathbb{N}$,
- $\mathbb{N}\langle 1\rangle = \mathbb{N}^{\mathbb{N}} = \mathcal{N}$,
- $\mathbb{N}\langle 2\rangle = \mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$,
- $\mathbb{N}\langle n+1\rangle = \mathscr{C}(\mathbb{N}\langle n\rangle, \mathbb{N})$,
- Also $\mathbb{N}\langle\alpha\rangle$ for countable ordinal α.

Base-complexity

Examples

- Countably-based $= \mathbb{N}\langle 0\rangle$-based,
- $\mathcal{O}(\mathcal{N})$ is $\mathbb{N}\langle 1\rangle$-based but not $\mathbb{N}\langle 0\rangle$-based,
- $\mathbb{N}\langle\alpha\rangle$ is $\mathbb{N}\langle\alpha+1\rangle$-based.

Questions

Is the base-complexity hierarchy proper?
What is the exact base-complexity of $\mathbb{N}\langle\alpha\rangle$?

Base-complexity

Examples

- Countably-based $= \mathbb{N}\langle 0\rangle$-based,
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Questions

Is the base-complexity hierarchy proper?
What is the exact base-complexity of $\mathbb{N}\langle\alpha\rangle$?

Theorem

For $\alpha \geq 2$, $\mathbb{N}\langle\alpha\rangle$ is not $\mathbb{N}\langle\alpha\rangle$-based. The hierarchy is proper.

Base-complexity

Theorem (Attempt)

If $h : \mathbf{Y} \rightrightarrows \mathbf{Y}$ *has no fixed-point, then* $\mathscr{C}(\mathbf{X}, \mathbf{Y})$ *is not a continuous image of* $\mathbf{X}$.

The diagonal argument does not work: it produces a *multi-valued* function $f : \mathbf{X} \rightrightarrows \mathbf{Y}$.

Base-complexity

If there exists $P \subseteq \mathbf{X}$ such that:

- $\mathbf{X}$ is a continuous image of P,

$$\begin{array}{c} \mathbf{X} \\ \uparrow \\ P \end{array}$$

Base-complexity

If there exists $P \subseteq \mathbf{X}$ such that:

- $\mathbf{X}$ is a continuous image of P,
- Every $f : P \to \mathbf{Y}$ has an extension $\overline{f} : \mathbf{X} \to \mathbf{Y}$,

$$\begin{array}{ccc} \mathbf{X} & \qquad & \mathscr{C}(\mathbf{X}, \mathbf{Y}) \\ \uparrow & & \downarrow \\ P & & \mathscr{C}(P, \mathbf{Y}) \end{array}$$

Base-complexity

If there exists $P \subseteq \mathbf{X}$ such that:

- $\mathbf{X}$ is a continuous image of P,
- Every $f : P \to \mathbf{Y}$ has an extension $\overline{f} : \mathbf{X} \to \mathbf{Y}$,
- And P embeds in $\mathcal{N}$,

$$\begin{array}{ccc} \mathbf{X} & & \mathscr{C}(\mathbf{X}, \mathbf{Y}) \\ \big\uparrow\!\!\uparrow & & \big\downarrow\!\!\downarrow \\ P & \xrightarrow{\text{no}}\!\!\!\!\rightarrow & \mathscr{C}(P, \mathbf{Y}) \end{array}$$

Base-complexity

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- Every $f : P \to \mathbf{Y}$ has an extension $\overline{f} : \mathbf{X} \to \mathbf{Y}$,
- And P embeds in $\mathcal{N}$,

then $\mathscr{C}(\mathbf{X}, \mathbf{Y})$ is not a continuous image of $\mathbf{X}$.

$$\begin{array}{ccc} \mathbf{X} & \overset{\text{no}}{\twoheadrightarrow} & \mathscr{C}(\mathbf{X}, \mathbf{Y}) \\ \uparrow\!\!\!\!\twoheadrightarrow & & \downarrow\!\!\!\!\twoheadrightarrow \\ P & \overset{\text{no}}{\twoheadrightarrow} & \mathscr{C}(P, \mathbf{Y}) \end{array}$$

Base-complexity

Theorem

For $\alpha \geq 2$, $\mathbb{N}\langle\alpha\rangle$ is not $\mathbb{N}\langle\alpha\rangle$-based.

Proof.

- $\mathbb{N}\langle\alpha\rangle$ contains such a P,
- $\mathcal{O}(\mathbb{N}\langle\alpha\rangle)$ has a fixed-point free multifunction, because $\mathbb{N}\langle\alpha\rangle$ is not countably-based. □

Base-complexity

Theorem

For $\alpha \geq 2$, $\mathbb{N}\langle\alpha\rangle$ *is not* $\mathbb{N}\langle\alpha\rangle$*-based.*

Proof.

- $\mathbb{N}\langle\alpha\rangle$ contains such a P,
- $\mathcal{O}(\mathbb{N}\langle\alpha\rangle)$ has a fixed-point free multifunction, because $\mathbb{N}\langle\alpha\rangle$ is not countably-based. □

Theorem

For $n \geq 2$, *there is no computable* $\boldsymbol{\Sigma}^1_n$*-indexing of the effective open sets of* $\mathbb{N}\langle n\rangle$.

Question

An analogy

- The $\mathbb{N}$-based spaces are the topological subspaces of $\mathcal{O}(\mathbb{N})$.
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[de Brecht, Schröder, Selivanov, 2016]

- For $\mathbb{N}$-based spaces:

$$\begin{aligned} \text{FPP} \iff \text{UFPP} &\iff \text{retract of } \mathcal{O}(\mathbb{N}) \\ &\iff \text{pointed } \omega\text{-continuous dcpo} \end{aligned}$$

- For $\mathcal{N}$-based spaces:

$$\begin{aligned} \text{FPP} \Longleftarrow \text{UFPP} &\iff \text{retract of } \mathcal{O}(\mathcal{N}) \\ &\iff ??? \end{aligned}$$

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