Rival-Sands principles in the Weihrauch degrees

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Joint work with Marta Fiori Carones and Paul Shafer

Rival-Sands theorem for graphs

Ivan Rival and Bill Sands. "On the Adjacency of Vertices to the Vertices of an Infinite Subgraph". In: *Journal of the London Mathematical Society* 2 (1980), pp. 393–400

Theorem (Rival-Sands)

RSg: *Let $G = (V, E)$ be an infinite countable graph. There is an infinite set $H \subseteq V$ such that*

- *for every $v \in V$, there are 0, 1 or infinitely many $h \in H$ such that $\{v, h\} \in E$.*
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Theorem (FC-Sh-So)

$\mathsf{RCA}_0 \vdash \mathsf{ACA}_0 \leftrightarrow \mathsf{RSg}$

Rival-Sands and RT^2_2

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Theorem (Fiori Carones-Hirst-Lempp-Shafer-Soldà)

$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{wRSgr} \leftrightarrow \mathsf{wRSg}$

Weihrauch reducibility et similia

We see every principle P as a **partial multifunction** $\mathsf{P} :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, and we describe it in terms of its valid **inputs** and of the valid **outputs** to those inputs.

Weihrauch reducibility: we say that $\mathsf{P} \leq_{\mathrm{W}} \mathsf{Q}$ if there are Turing functionals Φ, Ψ such that for every $x \in \mathrm{dom}(\mathsf{P})$, for every $y \in \mathsf{Q}(\Phi(x))$ it holds that $\Psi(x, y) \in \mathsf{P}(x)$.

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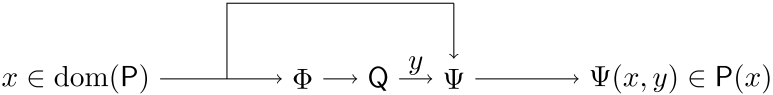
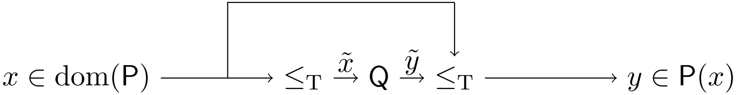


A related (weaker) concept is **computable reducibility**: $\mathsf{P} \le_\mathrm{c} \mathsf{Q}$ if for every $x \in \mathrm{dom}(\mathsf{P})$ there is a $\tilde{x} \le_\mathrm{T} x$ such that for every $\tilde{y} \in \mathsf{Q}(\tilde{x})$ there is $y \le_\mathrm{T} x \oplus \tilde{y}$ such that $y \in \mathsf{P}(x)$.



Weihrauch reducibility et similia (cont'd)

Another related (stronger) concept is **strong Weihrauch reducibility**: $\mathsf{P} \leq_{\mathrm{sW}} \mathsf{Q}$ if there are Turing functionals Φ, Ψ such that for every $x \in \mathrm{dom}(\mathsf{P})$ and for every $y \in \mathsf{Q}(\Phi(x))$, it holds that $\Psi(y) \in \mathsf{P}(x)$.

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- If $\mathrm{id} \times \mathsf{P} \leq_{\mathrm{sW}} \mathsf{P}$, then P is a **cylinder** and $\mathsf{Q} \leq_{\mathrm{W}} \mathsf{P}$ implies $\mathsf{Q} \leq_{\mathrm{sW}} \mathsf{P}$.

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Almost every principle that we are going to deal with today is a cylinder.

RSg in the Weihrauch degrees

Theorem (FC-Sh-So)

$\mathsf{RSg} \equiv_{\mathrm{sW}} \mathsf{WKL}''$

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RSg $\equiv_{\mathrm{sW}}$ WKL$''$

This has many consequences:

- RSg has universal instances.

  Recall: for a principle P, a **universal instance** is a computable input x^* such that for every $y^* \in \mathsf{P}(x^*)$ and any other computable instance x of P, there is $y \in \mathsf{P}(x)$ with $y \leq_{\mathrm{T}} y^*$.

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- A Turing degree $\mathbf{a}$ computes a RSg-solution to G if and only if $\mathbf{a}$ is PA relative to G''.
- Since WKL$'' \equiv_{\mathrm{W}} \widehat{\mathsf{RT}^2_2}$, as proved by Brattka and Rakotoniaina, "On the Uniform Computational Content of Ramsey's Theorem", RSg $\equiv_{\mathrm{W}} \widehat{\mathsf{RT}^2_2}$.

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- $\mathsf{RSg} \equiv_{\mathrm{sW}} \widehat{\mathsf{RSg}}$.
- RSg is effectively $\mathbf{\Sigma}^0_4$-measurable but not effectively $\mathbf{\Sigma}^0_3$-measurable.

Uniform strength of wRSg and wRSgr

- wRSg: for every infinite graph $G = (V, E)$, there is an infinite set $H \subseteq V$ such that for every $h \in H$, there are 0, 1, or infinitely many $h' \in H$ such that $\{h, h'\} \in E$.
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First of all, we could ask: what is the relationship between the two principles?

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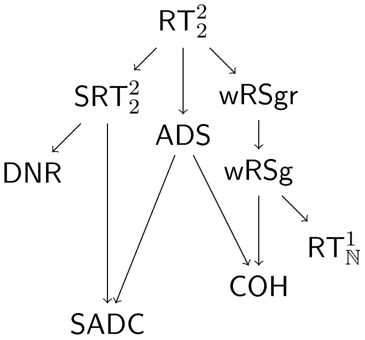
Question

Does wRSgr $\leq_{\mathrm{W}}$ wRSg hold?

wRSgr and wRSg in the Weihrauch degrees (spoiler alert)

- Astor et al., "The uniform content of partial and linear orders"
- Hirschfeldt and Shore, "Combinatorial principles weaker than Ramsey's theorem for pairs"
- Hirschfeldt, Jockusch, et al., "The Strength of Some Combinatorial Principles Related to Ramsey's Theorem for Pairs"
- Patey, "Partial Orders and Immunity in Reverse Mathematics"



In Fiori-Carones, Shafer, and Soldà, *An inside/outside Ramsey theorem and recursion theory*, only results about wRSgr and wRSg (and *maybe* $\mathsf{RT}^1_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{ADS}$).

Computability theoretic considerations

Lemma (FC-Sh-So)

If P *is a problem such that* $\mathsf{P} <_\omega \mathsf{RT}^2_2$, *then* $\mathsf{wRSg} \not\leq_{\mathrm{c}} \mathsf{P}$.

Recall: $\mathsf{P} \leq_\omega \mathsf{Q}$ if every ω-model of $\mathsf{RCA}_0 + \mathsf{Q}$ is a model of P.

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Theorem (FC-Sh-So)

For every infinite graph G*, there is a* wRSgr*-solution* H *to* G *such that* $H \leq_{\mathrm{T}} G'$.

Sketch of the proof: let $F := \{v \in V : v$ has finitely many neighbors$\}$.

- if F is finite, then $V \setminus F \leq_{\mathrm{T}} G$ is a solution.
- if F is infinite: since F is $\Sigma^{0,G}_2$, there is an infinite $\Delta^{0,G}_2$ $F_0 \subseteq F$. It is easy to computably find an infinite independent set $H \subseteq F_0$.

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Corollary (FC-Sh-So)

$\mathsf{RT}^2_2 \not\leq_{\mathrm{c}}$ wRSgr

What wRSg can do: COH and $\mathsf{RT}^1_{\mathbb{N}}$

Theorem (FC-Sh-So)

COH $\leq_{\mathrm{sW}}$ wRSg

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Sketch of the proof: we actually prove that CADS $\le_{\mathrm{W}}$ wRSg.

CADS	Input: an infinite linear order $(L, <_L)$ Output: an infinite $H \subseteq L$ such that for every $h \in H$, either $\{h' \in H : h' <_L h\}$ or $\{h' \in H : h' >_L h\}$ is finite

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3. $<_L$-increasingly enumerate points with at least two neighbors.

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1. if no $h \in H$ has 2 neighbors (in H), then $(H, <_L)$ is of type ω^*.

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2. if there is exactly one point h_0 with 2 (and so infinitely many) neighbors in H, then these points form a chain of type ω^* (actually, a *sequence* of that type). So $\Psi(H, (L, <_L))$ is a chain of type $1 + \omega^*$.

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2. if there is exactly one point h_0 with 2 (and so infinitely many) neighbors in H, then these points form a chain of type ω^* (actually, a *sequence* of that type). So $\Psi(H, (L, <_L))$ is a chain of type $1 + \omega^*$.
3. if there are $h_0 \neq h_1$ with at least 2 neighbors in H, then the set $\{h \in H : h \text{ has infinitely many neighbors in } H\}$ is infinite and has no $<_L$-maximum. So $\Psi(H, (L, <_L))$ is a chain of type $\omega + k$.

What wRSg can do: COH and $\mathsf{RT}^1_{\mathbb{N}}$ (cont'd)

The functional $\Psi(H,(L,<_L))$:

1. starts enumerating H, until an h_0 with 2 neighbors is found. Then
2. starts enumerating neighbors of h_0, until $h_1 \neq h_0$ with at least 2 neighbors is found. Then
3. $<_L$-increasingly enumerate points with at least two neighbors.

Why the construction works:

1. if no $h \in H$ has 2 neighbors (in H), then $(H,<_L)$ is of type ω^*.
2. if there is exactly one point h_0 with 2 (and so infinitely many) neighbors in H, then these points form a chain of type ω^* (actually, a *sequence* of that type). So $\Psi(H,(L,<_L))$ is a chain of type $1+\omega^*$.
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Finally, COH $\equiv_{\mathrm{sW}}$ CADS was proved by Hirschfeldt and Shore, "Combinatorial principles weaker than Ramsey's theorem for pairs".

What wRSg can do: COH and $\mathsf{RT}^1_{\mathbb{N}}$ (cont'd)

$\mathsf{RT}^1_{\mathbb{N}}$ Input: $f : \mathbb{N} \to \mathbb{N}$ with bounded range
Output: an infinite $H \subseteq \mathbb{N}$ such that $|f[H]| = 1$

What wRSg can do: COH and $\mathsf{RT}^1_{\mathbb{N}}$ (cont'd)

$\mathsf{RT}^1_{\mathbb{N}}$ Input: $f : \mathbb{N} \to \mathbb{N}$ with bounded range
Output: an infinite $H \subseteq \mathbb{N}$ such that $|f[H]| = 1$

$\mathsf{cRT}^1_{\mathbb{N}}$ Input: $f : \mathbb{N} \to \mathbb{N}$ with bounded range
Output: $i \in \mathbb{N}$ such that $f^{-1}(i)$ is infinite

What wRSg can do: COH and $\mathsf{RT}^1_{\mathbb{N}}$ (cont'd)

$\mathsf{RT}^1_{\mathbb{N}}$	Input: $f : \mathbb{N} \to \mathbb{N}$ with bounded range Output: an infinite $H \subseteq \mathbb{N}$ such that $\|f[H]\| = 1$
$\mathsf{cRT}^1_{\mathbb{N}}$	Input: $f : \mathbb{N} \to \mathbb{N}$ with bounded range Output: $i \in \mathbb{N}$ such that $f^{-1}(i)$ is infinite

Lemma (FC-Sh-So)

$\mathsf{RT}^1_{\mathbb{N}} \leq_{\mathrm{sW}}$ wRSg

Sketch of proof: Since $\mathsf{cRT}^1_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{RT}^1_{\mathbb{N}}$ and wRSg is a cylinder, it suffices to prove that $\mathsf{cRT}^1_{\mathbb{N}} \leq_{\mathrm{W}}$ wRSg.

Let $G_f = (\mathbb{N}, \{\{m,n\} \in [\mathbb{N}]^2 : f(m) = f(n)\})$, and let H be a wRSg-solution to G_f.

Let m be such that it has at least two neighbors in H. Output $f(m)$.

What wRSg can do: COH and $\mathsf{RT}^1_{\mathbb{N}}$ (cont'd)

$\mathsf{RT}^1_{\mathbb{N}}$ Input: $f : \mathbb{N} \to \mathbb{N}$ with bounded range
Output: an infinite $H \subseteq \mathbb{N}$ such that $|f[H]| = 1$

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Corollary (FC-Sh-So)

wRSg *and* wRSgr *are not parallelizable and are not effectively* $\mathbf{\Sigma}^0_2$*-measurable.*

A small digression: RT^1_k vs ADS

Lemma (FC-Sh-So)

$\mathsf{RT}^1_3 \leq_{\mathrm{sW}} \mathsf{ADS}$

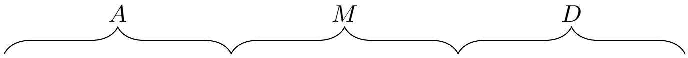
A small digression: RT^1_k vs ADS

Lemma (FC-Sh-So)

$\mathsf{RT}^1_3 \leq_{\mathrm{sW}} \mathsf{ADS}$

Sketch of proof: we show $\mathsf{cRT}^1_3 \leq_{\mathrm{W}} \mathsf{ADS}$. Let Φ be as follows:

c 0



$\Phi(c)$ 0

ascending color red, descending color undefined.

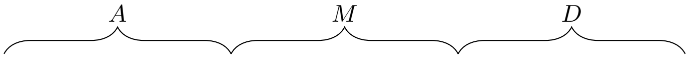
A small digression: RT^1_k vs ADS

Lemma (FC-Sh-So)

$\mathsf{RT}^1_3 \leq_{\mathrm{sW}} \mathsf{ADS}$

Sketch of proof: we show $\mathsf{cRT}^1_3 \leq_{\mathrm{W}} \mathsf{ADS}$. Let Φ be as follows:

c 0   1



$\Phi(c)$ 0 1

ascending color red, descending color blue.

A small digression: RT^1_k vs ADS

Lemma (FC-Sh-So)

$\mathsf{RT}^1_3 \leq_{\mathrm{sW}} \mathsf{ADS}$

Sketch of proof: we show $\mathsf{cRT}^1_3 \leq_{\mathrm{W}} \mathsf{ADS}$. Let Φ be as follows:

c	0	1	2
	A	M	D
$\Phi(c)$	2	0	1

ascending color green, descending color blue.

A small digression: RT^1_k vs ADS

Lemma (FC-Sh-So)

$\mathsf{RT}^1_3 \leq_{\mathrm{sW}} \mathsf{ADS}$

Sketch of proof: we show $\mathsf{cRT}^1_3 \leq_{\mathrm{W}} \mathsf{ADS}$. Let Φ be as follows:

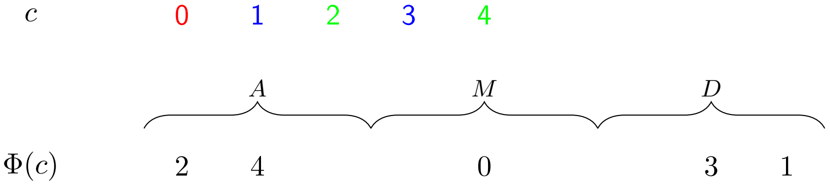
c	0	1	2	3				
		A			M		D	
$\Phi(c)$	2				0		3	1

ascending color green, descending color blue.

Lemma (FC-Sh-So)

$\mathsf{RT}^1_3 \leq_{\mathrm{sW}} \mathsf{ADS}$

Sketch of proof: we show $\mathsf{cRT}^1_3 \leq_{\mathrm{W}} \mathsf{ADS}$. Let Φ be as follows:



ascending color green, descending color blue.

A small digression: RT^1_k vs ADS

Lemma (FC-Sh-So)

$\mathsf{RT}^1_3 \leq_{\mathrm{sW}} \mathsf{ADS}$

Sketch of proof: we show $\mathsf{cRT}^1_3 \leq_{\mathrm{W}} \mathsf{ADS}$. Let Φ be as follows:

c	0	1	2	3	4	5	
		A			M		D
$\Phi(c)$	2	4		0	3	1	5

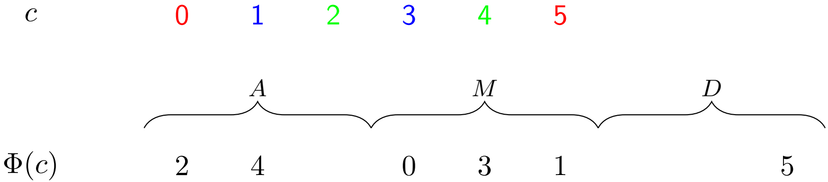
ascending color green, descending color red. Let H be an ADS-solution to $\Phi(c)$, and let $h_0 \in H$. Set $\Psi(c, H) = c(h_0)$.

A small digression: RT^1_k vs ADS

Lemma (FC-Sh-So)

$\mathsf{RT}^1_3 \le_{\mathrm{sW}} \mathsf{ADS}$

Sketch of proof: we show $\mathsf{cRT}^1_3 \le_{\mathrm{W}} \mathsf{ADS}$. Let Φ be as follows:



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Theorem (FC-Sh-So)

$\mathsf{RT}^1_5 \not\le_{\mathrm{W}} \mathsf{ADS}$

A small digression: RT^1_k vs ADS

Lemma (FC-Sh-So)

$\mathsf{RT}^1_3 \leq_{\mathrm{sW}} \mathsf{ADS}$

Theorem (FC-Sh-So)

$\mathsf{RT}^1_5 \not\leq_{\mathrm{W}} \mathsf{ADS}$

Question

Does $\mathsf{RT}^1_4 \leq_{\mathrm{W}} \mathsf{ADS}$ hold?

What wRSgr cannot do: SADC and DNR

Lemma (FC-Sh-So)

If $G = (V, E)$ is such that it has no wRSgr*-solution $H \leq_{\mathrm{T}} G$, then*

1. *G contains an infinite independent set.*

What wRSgr cannot do: SADC and DNR

Lemma (FC-Sh-So)

If $G = (V, E)$ *is such that it has no* wRSgr*-solution* $H \leq_{\mathrm{T}} G$*, then*

1. G *contains an infinite independent set.*
2. *For every finite independent* J *and cofinite* $\tilde{G} \subseteq G$*, there is* $H \subseteq \tilde{G}$ *such that* $J \cup H$ *is a* wRSgr*-solution to* G*.*

What wRSgr cannot do: SADC and DNR

Lemma (FC-Sh-So)

If $G = (V, E)$ is such that it has no wRSgr*-solution $H \leq_{\mathrm{T}} G$, then*

1. *G contains an infinite independent set.*
2. *For every finite independent J and cofinite $\tilde{G} \subseteq G$, there is $H \subseteq \tilde{G}$ such that $J \cup H$ is a* wRSgr*-solution to G.*

Theorem (FC-Sh-So)

- SADC $\not\leq_{\mathrm{W}}$ wRSgr
- DNR $\not\leq_{\mathrm{W}}$ wRSgr

SADC
Input: a stable infinite linear order $(L, <_L)$
Output: an infinite $H \subseteq L$ such that $(H, <_L)$ has type ω or ω^*

DNR
Input: $f : \mathbb{N} \to \mathbb{N}$
Output: a function $p : \mathbb{N} \to \mathbb{N}$ that is DNR with respect to f

What wRSgr cannot do: SADC and DNR (cont'd)

Suppose for a contradiction that SADC $\leq_{\mathrm{W}}$ wRSgr, with Φ and Ψ as witnesses, and let $(L, <_L)$ be a stable linear order without computable (and hence c.e.) ascending or descending chains.

What wRSgr cannot do: SADC and DNR (cont'd)

Suppose for a contradiction that SADC $\leq_{\mathrm{W}}$ wRSgr, with Φ and Ψ as witnesses, and let $(L, <_L)$ be a stable linear order without computable (and hence c.e.) ascending or descending chains.

If $G = (V, E)$ is such that it has no wRSgr-solution $H \leq_{\mathrm{T}} G$, then

1. G contains an infinite independent set.
2. For every finite independent J and cofinite $\tilde{G} \subseteq G$, there is $H \subseteq \tilde{G}$ such that $J \cup H$ is a wRSgr-solution to G.

$\Phi((L, <_L)) = (V, E)$ is an infinite graph. By 1 let C be an infinite independent set. Select an $x \in \Psi(C, (L, <_L))$, then there is a finite $D \subseteq C$ such that $x \in \Psi(D, (L, <_L))$.

What wRSgr cannot do: SADC and DNR (cont'd)

Suppose for a contradiction that SADC $\leq_{\mathrm{W}}$ wRSgr, with Φ and Ψ as witnesses, and let $(L, <_L)$ be a stable linear order without computable (and hence c.e.) ascending or descending chains.

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$$R := \{y \in L : \exists F \subset_{\text{fin}} V (F \text{ is independent and } \{x, y\} \subseteq \Psi(F, (L, <_L))\}.$$

What wRSgr cannot do: SADC and DNR (cont'd)

Suppose for a contradiction that SADC $\leq_{\mathrm{W}}$ wRSgr, with Φ and Ψ as witnesses, and let $(L, <_L)$ be a stable linear order without computable (and hence c.e.) ascending or descending chains.

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Since R is c.e., there is a $y \in R$ such that $[x, y]_L$ is infinite.

What wRSgr cannot do: SADC and DNR (cont'd)

Suppose for a contradiction that SADC $\leq_W$ wRSgr, with Φ and Ψ as witnesses, and let $(L, <_L)$ be a stable linear order without computable (and hence c.e.) ascending or descending chains.

If $G = (V, E)$ is such that it has no wRSgr-solution $H \leq_T G$, then

1. G contains an infinite independent set.
2. For every finite independent J and cofinite $\tilde{G} \subseteq G$, there is $H \subseteq \tilde{G}$ such that $J \cup H$ is a wRSgr-solution to G.

$\Phi((L, <_L)) = (V, E)$ is an infinite graph. By 1 let C be an infinite independent set. Select an $x \in \Psi(C, (L, <_L))$, then there is a finite $D \subseteq C$ such that $x \in \Psi(D, (L, <_L))$. We then define

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Let F be such that $\{x, y\} \subseteq \Psi(F, (L, <_L))$, by 2 we can extend it to a solution. Contradiction.

What wRSgr cannot do: SADC and DNR (cont'd)

Suppose for a contradiction that SADC $\leq_{\mathrm{W}}$ wRSgr, with Φ and Ψ as witnesses, and let $(L, <_L)$ be a stable linear order without computable (and hence c.e.) ascending or descending chains.

If $G = (V, E)$ is such that it has no wRSgr-solution $H \leq_{\mathrm{T}} G$, then

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DNR $\not\leq_{\mathrm{W}}$ wRSgr is proved similarly.

wRSg and wRSgr vs SRT^2_2

LPO Input: a function $f : \mathbb{N} \to \mathbb{N}$
Output: 0 if $f(n) = 0$ for some $n \in \mathbb{N}$, 1 otherwise

Lemma (FC-Sh-So)

1. $\mathsf{SRT}^2_2 \leq_{\mathrm{W}} \mathsf{LPO} * \mathsf{wRSgr}$
2. $\mathsf{SRT}^2_2 \leq_{\mathrm{W}} (\mathsf{LPO} \times \mathsf{LPO}) * \mathsf{wRSg}$

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Sketch of the proof of 1: given a stable $f : [\mathbb{N}]^2 \to 2$, let $G_f = (\mathbb{N}, \{\{n, s\} \in [\mathbb{N}]^2 : f(n, s) = 1\})$. Let H be a wRSgr-solution to G_f. We can use LPO to determine which case holds:

wRSg and wRSgr vs SRT^2_2

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- H is an independent set. Then H is an f-homogeneous set of color 0.

wRSg and wRSgr vs SRT^2_2

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- H is an independent set. Then H is an f-homogeneous set of color 0.
- There are adjacent $h, h' \in H$. Then H can be refined to an infinite homogeneous set of color 1.

Corollary (FC-Sh-So)

$\mathsf{RT}^2_2 \leq_{\mathrm{W}} \mathsf{wRSgr}^{[3]}$; $\mathsf{RT}^2_2 \leq_{\mathrm{W}} \mathsf{wRSg}^{[3]}$

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