Maximal order types of well partial orders

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Basics I

Well quasi order (wqo): not antisymmetric.

Well partial order (wpo)

A partial order in which there is no infinite descending sequence, and no infinite antichain.

Equivalently 'Bad sequences'

One in which there is no sequence $a_1, a_2, \ldots$ such that $i < j$ implies $a_i \not\leq a_j$.

Equivalently

One for which every linearization is a well-order.

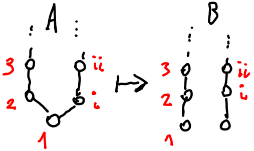
Basics II

Quasi-embedding

A map $f : A \to B$ such that $f(a_1) \leq f(a_2)$ implies $a_1 \leq a_2$.

Theorem

If B as above is wpo then A is wpo.



Since $B = \omega \sqcup \omega$, it is wpo.

Thus A is wpo.

Theorem

If C, D are wpo then so are $C \sqcup D$ and $C \times D$.

Higman ordering

The '*' constructor

$A^* = \{a_1 a_2 \cdots a_n : \forall i . a_i \in A\}$ is equipped with the ordering:

$$\sigma \leq \tau \text{ if } (\exists f : |\sigma| \to |\tau| \text{ increasing})(\forall i)\sigma(i) \leq \tau(f(i)).$$

Ex. In 3^*,

$111 \leq 222$

$11 \leq 21$

$11 \not\leq 2$

$1001 \leq 1000001$

Theorem (Higman's Lemma)

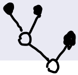
If A is wpo then A^* is wpo.

The constructor T 'Tree constructor'

$T(X \times X \sqcup \{\bullet\})$ 'Binary trees' — Strong embedding

Elts. $x = \bullet$
$x = o(x_1, x_2)$

Eg. $o(o(\bullet,\bullet),\bullet)$

$\approx$ 

$x \leq x'$ if $x = \bullet$
or $x = o(x_1, x_2)$ and $x' = o(x_1', x_2')$
and either $x \leq x_1'$ or $x \leq x_2'$
or $(x_1 \leq x_1'$ and $x_2 \leq x_2')$.

$T(A \times X \sqcup \{\bullet\})$ 'strings' — Higman ordering

Elts. $x = \bullet$
$x = o(a, x_1)$

E.g. $o(a_1, o(a_2, \dots o(a_n, \bullet)))$
$\approx a_1 \cdots a_n$

$x \leq x'$ if $x = \bullet$
or $x = o(a, x_1)$ and
$x' = o(a', x_1')$ and
either $x \leq x_1'$
or $a \leq a'$ and $x_1 \leq x_1'$

$T(X \sqcup \{\bullet\})$ $\mathbb{N}$ — usual ordering

Elts. $x = \bullet$
$x = o\, x_1$

E.g. $\underbrace{o\, o \cdots o}_{n \text{ times}} \bullet$
$\approx \overbrace{S\, S \cdots S}\, 0$

$x \leq x'$ if $x = \bullet$
or $(x = o x_1,\ x' = o x_1'$
and $x_1 \leq x_1')$

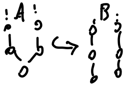
Basics III

Theorem (De Jongh, Parikh)

Let A be a wpo, and let mA be the supremum of the order types of all its linear extensions. Then there is a linear extension with order type mA.

Theorem

If $A \to B$ is a quasiembedding then $mA \leq mB$.



$mA \leq mB$

$= m(\omega \sqcup \omega)$

$= \omega \cdot 2$

$(\omega^2+1) \oplus \omega = \omega^2 + \omega + 1$

$\oplus, \otimes$ 'natural sum, product'

Theorem

$m(C \sqcup D) = mC \oplus mD$, and $m(C \times D) = mC \otimes mD$.

Basics IV

Left sets

If $a \in A$ then $L(a) = \{b \in A : a \not\leq b\}$. "Left set of a"

Theorem

A is wpo iff $L(a)$ is wpo for every $a \in A$.

Theorem

$$mA = \sup_{a \in A}(mLa + 1)$$

Old theorem, 'our proof'

In some sense implicit in thesis of Diana Schmidt 79.

Higman's Lemma

If A is wpo then A^* is wpo.

$A^* = T(A \times X \sqcup \{\bullet\})$. We need to show that Lx is w.p.o. for every x.
By induction.

If $x = \bullet$ then $Lx = \emptyset$. ✓

If $x = o(a, x_1)$ then what is Lx?

$x \not\leq x'$ if : $x' = \bullet$

or $x' = o(a', x_1')$
and either $x_1 \not\leq x_1'$
or $a \not\leq a'$ and $x \not\leq x_1'$.

$a\sigma \leq b\tau$
if ($a \leq b$ and $\sigma \leq \tau$)
or $a\sigma \leq \tau$

So $Lx \rightsquigarrow \{\bullet\} \sqcup A \times Lx_1 \sqcup La \times \underbrace{Lx}_{\substack{x_1', \text{ lower} \\ \text{complexity.}}}$

x'

$\rightsquigarrow T(La \times X \sqcup A \times Lx_1 \sqcup \{\bullet\})$

which is w.p.o. by induction on this 'ordinal polynomial'.

$mLa < mA$. ∎

$mLx \leq mT(La \times X \sqcup \sim)$

Old theorem 2

$m\,1^* = \omega \quad m\,2^* = \omega^\omega \quad m\,3^* = \omega^{\omega^2}$

Theorem

Let $\mathbb{B}$ denote the binary trees with strong embedding.
Then $\mathbb{B}$ is a wpo with $m\mathbb{B} \leq \epsilon_0$.

Pf. $\mathbb{B} = T(X \times X \sqcup \{\bullet\})$.

If $x = \bullet$ then $Lx = \emptyset$ ✓.

If $x = \circ(x_1, x_2)$ then

$$Lx \rightsquigarrow \{\bullet\} \sqcup \underline{Lx_1} \times Lx \sqcup \underline{Lx} \times Lx_2$$

lower complexity

$$\rightsquigarrow T((Lx_1 \sqcup Lx_2) \times X \sqcup \{\bullet\})$$
$$\simeq (Lx_1 \sqcup Lx_2)^*$$

Since this is wpo by ~~Higman~~. induction

So Lx is wpo.

Furthermore, by induction

$mLx_i < \varepsilon_0$

i.e. $< \omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}$

And by known results,

this means $m(Lx_1 \sqcup Lx_2)^* \leq$ about $\omega^{\omega^{(\omega^{\cdot^{\cdot^{\cdot^{\omega}}}})}}$

Weiermann's Conjecture

$k\beta =$ 'maximal coefficient'

$$\vartheta(\beta) = (\mu\gamma > \overset{k\beta}{\cancel{\beta}})(\forall\alpha < \overset{\beta}{\cancel{\gamma}})[k\alpha < \overset{\gamma}{\cancel{k\beta}} \implies \vartheta(\alpha) < \gamma]$$

$\Omega = \omega_1$.

Theorem

- $mT(X \cup \{\bullet\}) = \vartheta(\Omega)$ $\qquad T^\bullet(X) \qquad T(X)$
- $mT(X \times 17 \cup \{\bullet\}) = \vartheta(\Omega \cdot 17)$
- $mT(X \times X \cup \{\bullet\}) = \vartheta(\Omega^2) = \varepsilon_0$
- (van der Meeren) $T(B(X) \cup \{\bullet\}) = \vartheta(\varepsilon_{\Omega+1})$ $\qquad (\varepsilon_\Omega = \Omega)$
  $[T^\bullet(T^X(Y \times Y)) = \vartheta_0\vartheta_1(\Omega_1^2)]$

Conjecture

$mT(W(X)) = \vartheta(W(\Omega))$, always. Also for multiple T.

Thank you