Strong Minimal Pair Problem

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Outline

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Remarks on Priority Methods

Acknowledgement

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Minimal Pairs in R.E. Degrees

- We study the structure of recursively enumerable (r.e.) Turing degrees $(\mathcal{R}, \leq)$.
- Since Post [1944], many interesting results are obtained.
- **Theorem** (Lachlan and Yates) There are r.e. degrees $\mathbf{a}, \mathbf{b} > \mathbf{0}$ with infimum $\mathbf{0}$.
- Such a pair is called a *minimal pair*.

Strong Minimal Pairs

- An r.e. minimal pair $\mathbf{a}, \mathbf{b}$ is called *strong* if furthermore for any nonzero r.e. degree $\mathbf{x} \leq \mathbf{a}$, $\mathbf{b} \vee \mathbf{x} \geq \mathbf{a}$.
- Note: It is "one-sided". One also has a "two-sided" version.
- In fact, strong minimal pairs are closer to "Slaman triples" than "minimal pairs".

Slaman Handwritten Notes

Oct. 1983 ①

Theorem There are r.e. sets A, B and C and a Δ^0_2 set X so that

1. $0 < X \le A$
2. $C \not\le_T X \oplus B$
3. $\forall W \, (0 < W \le A \wedge W \text{ r.e.} \to C \le W \oplus B)$

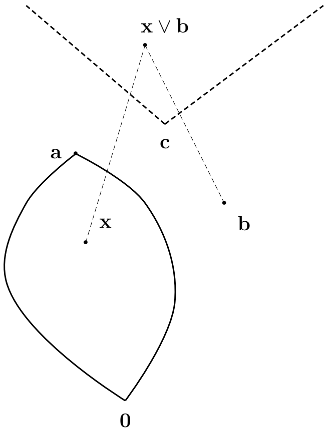
The theorem answers a question of Carl Jockusch by showing that the structure of the r.e. degrees with $\le_T$ is not a Σ_1 substructure of the Δ^0_2 degrees with $\le_T$. The Δ_0 formula with r.e. parameters satisfiable in the Δ^0_2 degrees but not in the r.e. degrees is

$$0 < x \le a \;\&\; x \le y \;\&\; b \le y \;\&\; c \not\le y$$

where a, b and c are the degrees of A, B and C respectively. There is no solution in the r.e. degrees

Slaman Triples

Theorem (Slaman) There are r.e. degrees $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ such that $\mathbf{a} > \mathbf{0}$, $\mathbf{c} \not\leq \mathbf{b}$, and for any nonzero r.e. degree $\mathbf{x} \leq \mathbf{a}$, $\mathbf{b} \vee \mathbf{x} \geq \mathbf{c}$.



- Strong Minimal Pair Problem says that one can make $\mathbf{a} = \mathbf{c}$.
- But in the "gap-cogap" construction of a Slaman Triple, $\mathbf{a}$ is on one side and $\mathbf{b}, \mathbf{c}$ is on another. It is not easy to merge $\mathbf{a}$ and $\mathbf{c}$.

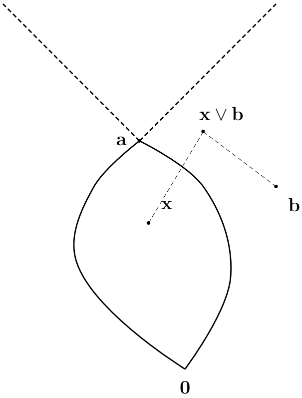
“A Long and Twisted History”

- In 2015, BCLS claimed that strong minimal pairs exist.
- Main Theorem of this talk: There are NO strong minimal pairs.
- We actually benefit greatly from BCLS's ideas and techniques, as we shall see later.
- BCLS: “[It] has a long and twisted history. It was discussed and claimed, in both directions, by a number of researchers over the past 25 years.”
- (Our starting point: BCLS asked if there is a two-sided strong minimal pair.)

Main Theorem

Theorem

For any r.e. degrees $\mathbf{a}$ *and* $\mathbf{b}$, *either* $\mathbf{a} \leq \mathbf{b}$ *or there exists an r.e. degree* $\mathbf{x} \leq \mathbf{a}$ *such that* $\mathbf{x} \neq \mathbf{0}$ *and* $\mathbf{x} \vee \mathbf{b} \not\geq \mathbf{a}$.



Corollary

Corollary

If $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ *form a Slaman triple, then* $\mathbf{a}$ *and* $\mathbf{c}$ *form a minimal pair.*

Otherwise, any nonzero $\mathbf{x} \leq \mathbf{a}, \mathbf{c}$ and $\mathbf{b}$ form a strong minimal pair.

(This fits the gap-cogap construction.)

Highlights (I): Nonuniformity

- BCLS spent section 3 illustrating how to defeat two requirements/candidates and their idea works.
- However, they overlooked the more complicated interactions of three requirements, which cause new problems.
- Our approach: We need an r.e. set X s.t. $X \leq_T A$ and satisfies the following requirements:
  - G_e: (assuming $A \nleq B$) $\Gamma_e(B \oplus X) \neq A$; and
  - D_e: (assuming $A \nleq 0$) $\Delta_e \neq X$.
- In fact, we built one U, a family V_α and families $W_{\alpha,\beta}$ so that one of them will play the role of X.

Nonuniformity (conti.)

- **Q**: Is it necessary?
- BCLS can defeat two candidates explicitly. Their method seemed to fail when 3 candidates are "active".
- (In fact, for fixed n, there seemed a variation of BCLS to organize the requirements, so that we can defeat n sets.)
- That is the reason we exploit 3 families of candidates.

Highlights (II): $\omega + 1$-Branching

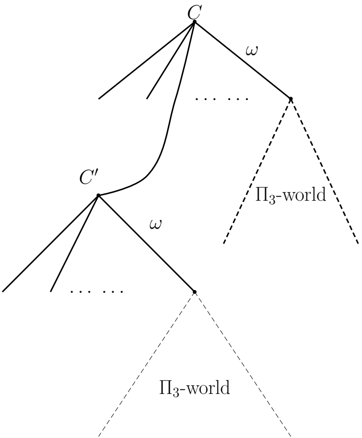
- Inspired by BCLS's "*c*-outcome", we have certain C-nodes whose outcomes are arranged with order type $\omega + 1$.

$$0 <_L 1 <_L 2 <_L \cdots <_L \omega.$$

- The outcome $C\hat{\ }n$ indicates that $\Gamma(BU; n)\uparrow$, which is a Σ_3-outcome; whereas $C\hat{\ }\omega$ indicates $\Gamma(BU)$ is total, which is a Π_3-outcome.
- It was used long before. E.g. in Shore's Nonjump-Inversion Theorem (1988).
- We view $C\hat{\ }\omega$ as the gateway to a Π_3-*world*, which is parallel to Σ_3-world (those below the $C\hat{\ }n$'s).

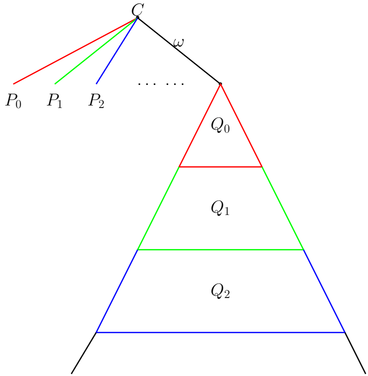
Highlights (II) (conti.): Π_3-Worlds

- Below $C^\frown n$, there might be other C' which leads to another Π_3-world.
- Once we enter a Π_3-world, we assume $\Gamma(BU)$ is total and U is finished. There will be no more Π_3-worlds, i.e., there is no nesting of Π_3-worlds.
- (Again this is not new.)



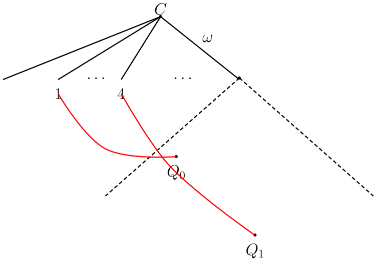
Π_3-worlds (conti.): Static Priority

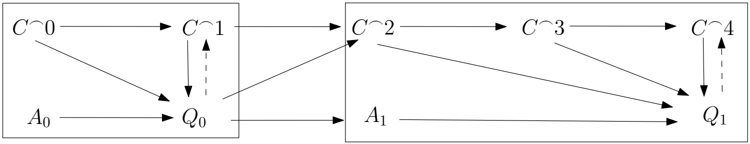
Usually, we can partition the Π_3-world into ω pieces and interlace them with $\{C\hat{\ }n : n \in \omega\}$ to get priority.



Π_3-worlds (conti.): Dynamic Priority

- Pairing at one C-node: At each stage, we may pair some $C\hat{\ }n_i$ in Σ_3-world with some nodes Q_i in Π_3-world. And determine priority "accordingly".
- (We actually had a partial order.)





Π_3-worlds (conti.): More Dynamic Priority

- We may have some nodes $P_k \prec C\hat{\ } n_i$ who may attack U and nodes Q_j may attack V.
- Like in the minimal pair, we don't want U and V to change at the same time. *Dynamic priority*: The side who acts first may injure the other side.
- This adds more burdens on showing the existence of true path.

True Path

- We define the true path to be the left-most one which is visited infinitely often.
- (A strange feature: We can't rule out that both Σ_3-world and Π_3-world(s) contain a true path, although our definition favors the Σ_3-one.)
- As pointed out in Shore [1988], it takes $\mathbf{0}'''$ to figure out the true path on $(\omega + 1)$-setting.
- Due to nonuniformity, it takes $\mathbf{0}^{(4)}$ to figure out which set of U, V_α and $W_{\alpha,\beta}$ wins.

Using $\mathbf{0}^{(4)}$ to Determine the Winner

Lemma

Given r.e. sets $A = W_a$ and $B = W_b$ with $A \nleq_T B$, there is a function $f \leq_T \varnothing^{(4)}$ such that $\varnothing <_T W_{f(a,b)} \leq_T A$ and $A \nleq_T B \oplus W_{f(a,b)}$.

Sketch: We use $\varnothing^{(4)}$ to decide if $A \leq_T B$. If $A \leq_T B$, define $f(a,b) = 0$. If $A \nleq_T B$, one can show that ρ is infinite and $\rho \leq_T \varnothing'''$.

Next, we use $\rho' \leq_T \varnothing^{(4)}$ to decide whether ρ ever enters a Π_3-world.

Case 1. Yes, it enters. Then use the unique C-node as a parameter, ρ becomes $\leq \varnothing''$. Ask ρ'' if there are infinitely many V-nodes on ρ, if yes, then V wins; otherwise the "last" W wins.

Case 2. No, it doesn't. Then again ρ becomes $\leq \varnothing''$. Ask ρ'' if there are infinitely many U-nodes, if yes, then U wins; otherwise do as in Case 1 after the "last" U.

Is It Necessary?

Using BCLS, we can show that

Theorem

Given $f \leq_T \varnothing'''$, we can find two nonrecursive incomparable r.e. sets $A = W_a$ and $B = W_b$ such that for $e = f(a,b)$, either $W_e \nleq_T A$, or $W_e \equiv \varnothing$ or $W_e \oplus B \geq_T A$.

- We outline a proof for the weaker version which ruled out $f \leq_T \varnothing''$.
- By Recursion Theorem, we know the indices a, b we are constructing. f is recursive in $\varnothing''$, so the relation $e = f(a,b)$ is both Σ_3 and Π_3. Let R and S be Π_2 predicates such that:

$$e = f(a,b) \iff \exists x R(x,a,b,e) \iff \neg \exists y S(y,a,b,e) \quad (1)$$

Proof (conti.)

We use a priority tree to guess the value of Π_2 predicates.

The root is labelled G_0 (G for guessing),which tests whether $0 = f(a,b)$. This is done by finding the correct witness x or y for $R(x,a,b,0)$ or $S(y,a,b,0)$. G_0 has ω many outcomes labeled by $n \in \omega$. The even ones $2x$ are testing for each x and the odd ones $2y+1$ are testing for each y.

In the end, the true path gives us the correct witness x or y. If it is even, then we know that $0 = f(a,b)$ and have the correct index to work with below G_0, i.e., we use BCLS to attack W_0. (Since there is only one set W being involved, BCLS works.) If it is odd, then we continue with another guessing node G_1, which tests whether $1 = f(a,b)$, and so on. Eventually we can figure out the correct index along the true path and win.

Priority Methods

We are also interested in the proof method involved.

The intuitive complexity of priority methods is measured by:

- Counting (obviously too crude)
- Harrington's *Syntactical Analysis* of the requirements. Tension between recursive construction and the approximation of the Σ_n-instructions.
- (?) The function which provides the indices of the winning sets.
- The induction required (but that is more of reversing the theorem, not the construction).
- Lempp and Lerman's general framework.

Remarks on $\mathbf{0}^{(3)}$-arguments

- Finite and infinite injury methods are well-understood.
- Most of $\mathbf{0}'''$ injury method is viewed as finite injury on the true path (i.e., $\mathbf{0}'$ argument over $\mathbf{0}''$-one).
- Shore [1988]: "Our construction [non-jump inversion] is a $\mathbf{0}^{(3)}$ one for a different reason. We use an $(\omega+1)$-branching tree of strategies. Thus it takes $\mathbf{0}^{(3)}$ to simply calculate the true path in this tree. ... it might best be described as a $\mathbf{0}''$ argument over $\mathbf{0}'$."

Some Questions

- Are there other interesting types of $\mathbf{0}'''$ arguments?
- What is a typical $\mathbf{0}^{(4)}$-one?
- Is there a general framework which can prevent/minimize the risk of mistakes?