Arithmetic under negated induction

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> [T]he method is extremely valuable when we want to beat a particular theory into the ground. When it can be carried out, the method of elimination of quantifiers gives a tremendous amount of information about a theory.
>
> Chang–Keisler 1973

This talk

Main theorem

The theory $\mathsf{WKL}_0^* + \neg \mathsf{I}\Sigma_1^Z$ in the language for second-order arithmetic eliminates set quantifiers.

Plan

1. induction and collection
2. the Weak König Lemma
3. quantifier elimination
4. consequences

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First-order arithmetic

$n \in \mathbb{N}$

- $\mathcal{L}_1 = \{0, 1, +, \times, <, =\}$.
- A quantifier is *bounded* if it is of the form $\forall v < t$ or $\exists v < t$.
- An $\mathcal{L}_1$ formula is Δ_0 if all its quantifiers are bounded.
- $\Sigma_n = \{\exists \bar{v}_1\ \forall \bar{v}_2\ \cdots\ \mathrm{Q}\bar{v}_n\ \theta : \theta \in \Delta_0\}$ and $\Pi_n = \{\forall \bar{v}_1\ \exists \bar{v}_2\ \cdots\ \mathrm{Q}'\bar{v}_n\ \theta : \theta \in \Delta_0\}$.
- $\mathrm{I}\Sigma_n$ consists of the axioms of PA^- and for every $\theta \in \Sigma_n$,

$$\theta(0) \wedge \forall x\ \big(\theta(x) \rightarrow \theta(x+1)\big) \rightarrow \forall x\ \theta(x).$$

- $\mathrm{PA} = \bigcup_{k \in \mathbb{N}} \mathrm{I}\Sigma_k$.
- exp asserts the totality of $x \mapsto 2^x$ over $\mathrm{I}\Sigma_0$.
- $\mathrm{B}\Sigma_n$ consists of the axioms of $\mathrm{I}\Sigma_0$ and for every $\theta \in \Sigma_n$,

$$\forall a\ \big(\forall x < a\ \ \exists y\ \theta(x, y) \rightarrow \exists b\ \forall x < a\ \ \exists y < b\ \ \theta(x, y)\big).$$

Theorem (Paris–Kirby 1978)

$\mathrm{I}\Sigma_0 + \exp \dashv \mathrm{B}\Sigma_1 + \exp \dashv \mathrm{I}\Sigma_1 \dashv \mathrm{B}\Sigma_2 \dashv \mathrm{I}\Sigma_2 \dashv \mathrm{B}\Sigma_3 \dashv \mathrm{I}\Sigma_3 \dashv \mathrm{B}\Sigma_4 \dashv \mathrm{I}\Sigma_4 \dashv \cdots$ and none of the converses holds.

Model theory of fragments of PA

$n \in \mathbb{N}$

Insight (Kaye, around 1991)

The model-theoretic properties of a model of arithmetic do not only depend on the induction axioms it satisfies, but also on the induction axioms it does *not* satisfy.

Theorem (Kossak 1990, Kaye 1991)

Every countable model of $\mathrm{B}\Sigma_{n+1} + \exp + \neg\mathrm{I}\Sigma_{n+1}$ has $2^{\aleph_0}$-many automorphisms and proper elementary cofinal substructures.

Theorem (Paris–Kirby 1978)

There is a countable model of $\mathrm{I}\Sigma_n + \exp + \neg\mathrm{B}\Sigma_{n+1}$ with no non-trivial automorphism and no proper elementary substructure.

Theorem (Paris–Kirby 1978)

$\mathrm{I}\Sigma_0 + \exp \dashv \mathrm{B}\Sigma_1 + \exp \dashv \mathrm{I}\Sigma_1 \dashv \mathrm{B}\Sigma_2 \dashv \mathrm{I}\Sigma_2 \dashv \mathrm{B}\Sigma_3 \dashv \mathrm{I}\Sigma_3 \dashv \mathrm{B}\Sigma_4 \dashv \mathrm{I}\Sigma_4 \dashv \cdots$ and none of the converses holds.

ω-extensions

$n \in \mathbb{N}$

Definition

An *ω-extension* of an $\mathcal{L}_2$ structure is an extension with no new number.

Theorem (Towsner 2015 for $n \geqslant 1$)

Given any countable $(M, \mathscr{X}) \models \mathrm{I}\Sigma^0_n + \exp + \neg\mathrm{B}\Sigma^0_{n+1}$ and any $S \subseteq M$, one can ω-extend $(M, \mathscr{X})$ to $(M, \mathscr{Y}) \models \mathrm{I}\Sigma^0_n + \exp + \neg\mathrm{B}\Sigma^0_{n+1}$ in which S is definable.

Proposition

For every countable $(M, \mathscr{X}) \models \mathrm{B}\Sigma^0_{n+1} + \exp + \neg\mathrm{I}\Sigma^0_{n+1}$, there is $S \subseteq M$ such that one can *never* ω-extend $(M, \mathscr{X})$ to $(M, \mathscr{Y}) \models \mathrm{B}\Sigma^0_{n+1} + \exp + \neg\mathrm{I}\Sigma^0_{n+1}$ in which S is definable.

Theorem (Paris–Kirby 1978)

$\mathrm{I}\Sigma^0_0 + \exp \dashv \mathrm{B}\Sigma^0_1 + \exp \dashv \mathrm{I}\Sigma^0_1 \dashv \mathrm{B}\Sigma^0_2 \dashv \mathrm{I}\Sigma^0_2 \dashv \mathrm{B}\Sigma^0_3 \dashv \mathrm{I}\Sigma^0_3 \dashv \mathrm{B}\Sigma^0_4 \dashv \mathrm{I}\Sigma^0_4 \dashv \cdots$ and none of the converses holds.

Preservation theorem

We identify $\mathcal{L}_2$ formulas with their Gödel numbers.

Definition

An *ω-extension* of an $\mathcal{L}_2$ structure is an extension with no new number.

Lemma (elementary)

Let T, T^* be $\mathcal{L}_2$ theories, where T is Π^1_1-axiomatized. If every countable [recursively saturated] model of T with finitely many sets has an ω-extension to a model of T^*, then $T \vdash \Pi^1_1\text{-Th}(T^*)$.

Definition

A type $p(\bar{v}, \bar{V})$ over an $\mathcal{L}_2$ structure $(M, \mathscr{X})$ is *recursive* if it involves only finitely many free variables and finitely many parameters $\bar{c}, \bar{C} \in (M, \mathscr{X})$, and

$$\{\theta(\bar{v}, \bar{V}, \bar{z}, \bar{Z}) : \theta(\bar{v}, \bar{V}, \bar{c}, \bar{C}) \in p(\bar{v}, \bar{V})\}$$

is recursive. The structure is *recursively saturated* if it realizes all recursive types.

Theorem (mostly Barwise 1975, Ressayre 1977, independently)

Let T, T^* be $\mathcal{L}_2$ theories, where T is Π^1_1-axiomatized and T^* is recursively axiomatized. If $T \vdash \Pi^1_1\text{-Th}(T^*)$, then every countable *recursively saturated* model of T with finitely many sets has an ω-extension to a model of T^*.

Ramsey's Theorem for pairs for two colours

Definition

RT^2_2 denotes an $\mathcal{L}_2$ sentence which expresses "whenever each unordered pair of numbers is given exactly one of two colours, there is an unbounded monochromatic set" over $\mathsf{I}\Sigma_0 + \exp$.

Open question

Does $\mathsf{B}\Sigma^0_2 \vdash \Pi^1_1\text{-Th}(\mathsf{RCA}_0 + \mathsf{RT}^2_2)$?

$\mathsf{RCA}_0 = \mathsf{I}\Sigma^0_1 + \Delta^0_1$-comprehension.

Model-theoretic version of the question

Does every countable recursively saturated $(M, \mathscr{X}) \models \mathsf{B}\Sigma^0_2$ with finitely many sets have an ω-extension to a model of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$?

Partial answer (Cholak–Jockusch–Slaman 2001)

Yes, if $(M, \mathscr{X}) \models \mathsf{I}\Sigma^0_2$.

Remaining question

Does every countable recursively saturated $(M, \mathscr{X}) \models \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$ with finitely many sets have an ω-extension to a model of $\mathsf{RCA}_0 + \mathsf{RT}^2_2$?

The Weak König Lemma

- $\mathrm{B}\Sigma_1 + \exp$ and WKL_0^* are respectively the first- and the second-order theories of exponentially closed initial segments in models of arithmetic.
- WKL_0^* consists of $\mathrm{I}\Sigma_0^0 + \exp$, the Δ_1^0 comprehension scheme, and an axiom stating "every unbounded 0–1 tree has an unbounded path".

Any set of numbers that is both Σ_1^0- and Π_1^0-definable is in the set universe.

Proposition (Simpson–Smith 1986)

$\mathrm{WKL}_0^* \vdash \mathrm{B}\Sigma_1^0 + \exp$.

Theorem (Simpson–Smith 1986)

Every countable model of $\mathrm{B}\Sigma_1^0 + \exp$ has an ω-extension to a model of WKL_0^*.

not necessarily recursively saturated

Proof

Force in the style of Jockusch–Soare (1972), where conditions are unbounded trees. □

Corollary (Simpson–Smith 1986)

$\mathrm{B}\Sigma_1^0 + \exp$ axiomatizes Π_1^1-$\mathrm{Th}(\mathrm{WKL}_0^*)$.

The model-theoretic core

$\forall b\ \exists w < a\ \forall v < b\ \ \dots$
$\leftrightarrow \exists w < a\ \forall v\ \ \dots$

Theorem

Every countable $(M, \mathscr{X}) \models \mathrm{B}\Sigma^0_1 + \exp + \neg\mathrm{I}\Sigma^0_1$ has a unique countable ω-extension $(M, \mathscr{Y}) \models \mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ up to isomorphism.

$\forall b\ \exists P\ \forall v < b\ \ \dots\ \ \leftrightarrow \exists P\ \forall v\ \ \dots$

Proof sketch

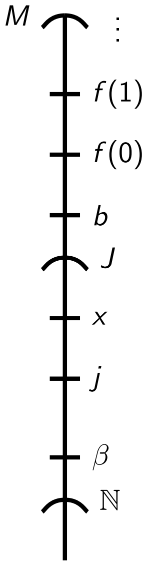
Given two such extensions $(M, \mathscr{Y}), (M, \mathscr{Z})$, build an isomorphism between them by a back-and-forth construction.

At every stage, we have $\bar{r}, \bar{R} \in (M, \mathscr{Y})$ and $\bar{s}, \bar{S} \in (M, \mathscr{Z})$ such that

$$\exists \beta \in M \setminus \mathbb{N} \quad \exists b \in M \setminus J \quad \forall\ \Sigma^0_0 \text{ formula } \theta < \beta \quad \forall \bar{x} < b \quad \forall \bar{j} \in J$$
$$(M, \mathscr{Y}) \models \theta(\bar{x}, f(\bar{j}), \bar{r}, \bar{R}, A) \quad \Leftrightarrow \quad (M, \mathscr{Z}) \models \theta(\bar{x}, f(\bar{j}), \bar{s}, \bar{S}, A),$$

where

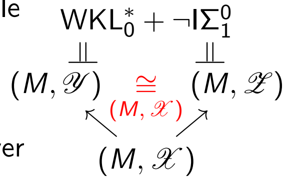
- J is a proper initial segment of M that is closed under $x \mapsto 2^x$ and is Σ^0_1-definable over the parameter $A \in \mathscr{X}$ in $(M, \mathscr{X})$; and
- $f : J \to M$ whose graph is Σ^0_0-definable over A in $(M, \mathscr{X})$ and whose range is cofinal in M. □



Quantifier elimination

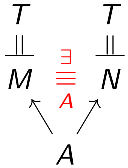
Theorem

Every countable $(M, \mathscr{X}) \models \mathsf{B}\Sigma^0_1 + \exp + \neg\mathsf{I}\Sigma^0_1$ has a unique countable ω-extension $(M, \mathscr{Y}) \models \mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1$ up to isomorphism.



Lemma (folklore?)

A theory T has quantifier elimination if the following is true: whenever A is a common substructure of $M, N \models T$, if $\bar{a} \in A$ such that $M \models \exists y\ \varphi(\bar{a}, y)$, where φ is quantifier-free, then $N \models \exists y\ \varphi(\bar{a}, y)$.



Main theorem

Every $\mathcal{L}_2$ formula is equivalent to a Δ^1_0 formula over $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^Z_1$.

Proof

Run a proof of the lemma above on countable recursively saturated models, so that all extensions can be assumed to be ω-extensions. $\square$

Over $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^Z_1$, every $\mathcal{L}_2$ formula $\theta(\bar{x}, \bar{Y})$ is equivalent to $\mathbb{JS}_{\bar{Y},Z} \Vdash \theta(\check{\bar{x}}, \check{\bar{Y}})$.

The Weak König Lemma as a model completion

Main theorem

Every $\mathcal{L}_2$ formula is equivalent to a Δ^1_0 formula over $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^Z_1$.

Corollary

$\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^Z_1$ is the unique $\mathcal{L}_2$ theory T such that

(a) $\Pi^1_1\text{-Th}(T) = \Pi^1_1\text{-Th}(\mathsf{B}\Sigma^0_1 + \exp + \neg\mathsf{I}\Sigma^Z_1)$; and

(b) every Π^1_1 formula is equivalent to a Σ^1_1 formula over T.

$\Delta^1_0 \mapsto$ quantifier-free

$\Pi^1_1 \mapsto \forall_1$

$\Pi^1_2 \mapsto \forall_2$

Theorem (Simpson 1999, after Kleene)

Provably in ACA_0, some Π^1_1 formula is not equivalent to any Σ^1_1 formula. In particular, this fact is true in $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.

Corollary

Models of $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^Z_1$ are precisely the Σ^1_1-*closed* models of $\mathsf{B}\Sigma^0_1 + \exp + \neg\mathsf{I}\Sigma^Z_1$, i.e., if a Σ^1_1 formula, possibly with parameters, can be satisfied in a Δ^1_0-elementary extension satisfying $\mathsf{B}\Sigma^0_1 + \exp + \neg\mathsf{I}\Sigma^Z_1$, then it is already true in the ground model.

Π^1_1 conservativity over $\mathsf{B}\Sigma^0_1 + \exp + \neg\mathsf{I}\Sigma^0_1$

σ is *Π^1_1-conservative* over $\mathsf{B}\Sigma^0_1 + \exp + \neg\mathsf{I}\Sigma^0_1$.

Corollary

The following are equivalent for all Π^1_2 sentences σ.

(i) $\Pi^1_1\text{-Th}(\mathsf{B}\Sigma^0_1 + \exp + \neg\mathsf{I}\Sigma^0_1 + \sigma) = \Pi^1_1\text{-Th}(\mathsf{B}\Sigma^0_1 + \exp + \neg\mathsf{I}\Sigma^0_1)$.

(ii) $\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1 \vdash \sigma$.

(iii) Every countable model of $\mathsf{B}\Sigma^0_1 + \exp + \neg\mathsf{I}\Sigma^0_1$ has an ω-extension satisfying $\mathsf{B}\Sigma^0_1 + \exp + \neg\mathsf{I}\Sigma^0_1 + \sigma$.

WKL^*_0 is the strongest Π^1_1-conservative Π^1_2 sentence over $\mathsf{B}\Sigma^0_1 + \exp + \neg\mathsf{I}\Sigma^0_1$

Theorem

$\mathsf{WKL}^*_0 + \neg\mathsf{I}\Sigma^0_1 \nvdash \mathsf{RT}^2_2$.

Turing-equivalent to $\Pi_2\text{-Th}(\mathbb{N})$

Theorem

$\{\sigma \in \Pi^1_2\text{-Snt} : \Pi^1_1\text{-Th}(\mathsf{B}\Sigma^0_1 + \exp + \sigma) = \Pi^1_1\text{-Th}(\mathsf{B}\Sigma^0_1 + \exp)\}$ is Π_2-complete.

Proposition

There is a Π^1_2 sentence σ such that $\Pi^1_1\text{-Th}(\mathsf{B}\Sigma^0_1 + \exp + \sigma) = \Pi^1_1\text{-Th}(\mathsf{B}\Sigma^0_1 + \exp)$ but some countable model of $\mathsf{B}\Sigma^0_1 + \exp$ does not ω-extend to any model of $\mathsf{B}\Sigma^0_1 + \exp + \sigma$.

Pigeonhole Principles

$n \in \mathbb{N}$

In a model $M \models \mathsf{I}\Sigma_n + \exp + \neg\mathsf{B}\Sigma_{n+1}$,

(Dimitracopoulos–Paris 1986) for some $b \in M$, there is a Σ_{n+1}-definable injection $[0, b+1) \to [0, b)$;

(Groszek–Slaman 1994) *maybe* there is a Σ_{n+1}-definable bijection $M \to \mathbb{N}$;

(Belanger–Chong–Wang–W–Yang 2021) *maybe*, for every non-zero $b \in M$, there is *no* Σ_{n+1}-definable injection $[0, 2b) \to [0, b)$.

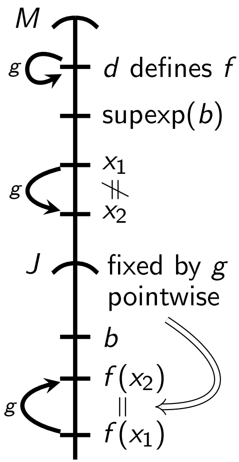
In a model $M \models \mathsf{B}\Sigma_{n+1} + \exp + \neg\mathsf{I}\Sigma_{n+1}$,

(Dimitracopoulos–Paris 1986) for some $b \in M$, there is a $(\Sigma_{n+1} \vee \Pi_{n+1})$-definable injection $[0, b+1) \to [0, b)$;

(Belanger–Chong–Li–W–Yang, in progress) *maybe*, for every $b \geqslant 2$ in M, there is *no* Σ_{n+3}-definable injection $[0, b^2) \to [0, b)$;

(Kołodziejczyk–Kowalik–Yokoyama 2021+) for every sufficiently large $b \in M$, there is *no* definable injection $f\colon [0, \mathrm{supexp}(b)) \to [0, b)$.

build $g \in \mathrm{Aut}(M)$



∴ f is not injective

The model theory of $\mathrm{B}\Sigma^0_1 + \exp + \neg\mathrm{I}\Sigma^0_1$

$n \in \mathbb{N}$

Main theorem

Every $\mathcal{L}_2$ formula is equivalent to a Δ^1_0 formula over $\mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^Z_1$.

Model-theoretic core

Every countable $(M, \mathscr{X}) \models \mathrm{B}\Sigma^0_1 + \exp + \neg\mathrm{I}\Sigma^0_1$ has a unique countable ω-extension $(M, \mathscr{Y}) \models \mathrm{WKL}^*_0 + \neg\mathrm{I}\Sigma^0_1$ up to isomorphism.

WKL^*_0 is the strongest Π^1_1-conservative Π^1_2 sentence.

- These results relativize to $\mathrm{B}\Sigma^0_{n+1} + \exp + \neg\mathrm{I}\Sigma^0_{n+1}$.
- The model theory of $\mathrm{I}\Sigma^0_n + \exp + \neg\mathrm{B}\Sigma^0_{n+1}$ is radically different.
- Such model-theoretic properties cannot be achieved without including a false-in-$\mathbb{N}$ sentence in the theory.

$\vdots$

$\top$

$\mathrm{I}\Sigma^0_2$

$\top$

$\mathrm{B}\Sigma^0_2$

$\top$

$\mathrm{I}\Sigma^0_1$

$\top$

$\mathrm{B}\Sigma^0_1 + \exp$

$\top$

$\mathrm{I}\Sigma^0_0 + \exp$

Insight (Kaye, around 1991)

The model-theoretic properties of a model of arithmetic do not only depend on the induction axioms it satisfies, but also on the induction axioms it does *not* satisfy.